Real-Time Integration of Lagrangian Sensors and Traffic Flow Models

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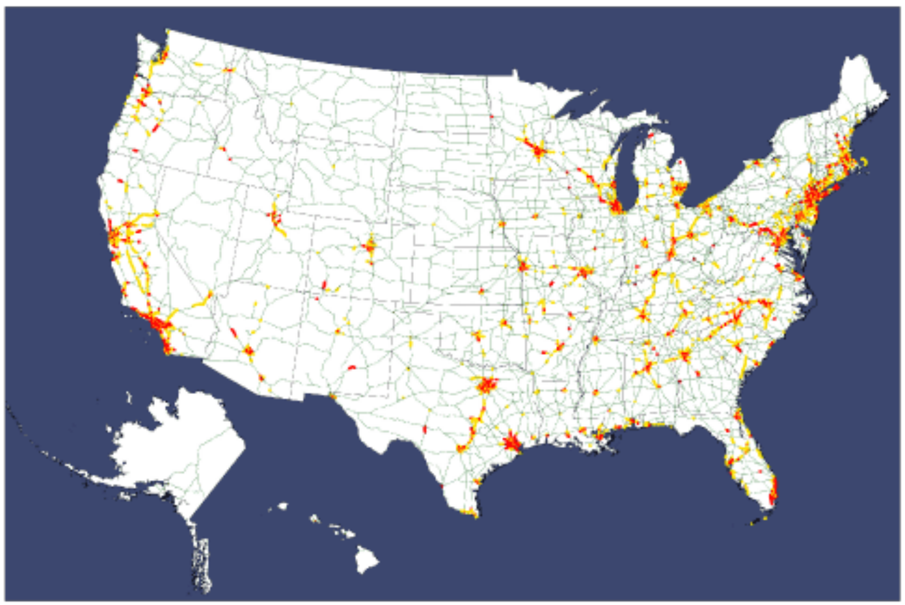
Road traffic congestion

- Congestion in the US in 2009 (Urban Mobility Report, 2010)
  - $115 billion in wasted time and fuel
  - 4.8 billion hours of delay
  - Average traveler needs 25% more time than speed limit travel-time

- Federal Highway Admistration trend



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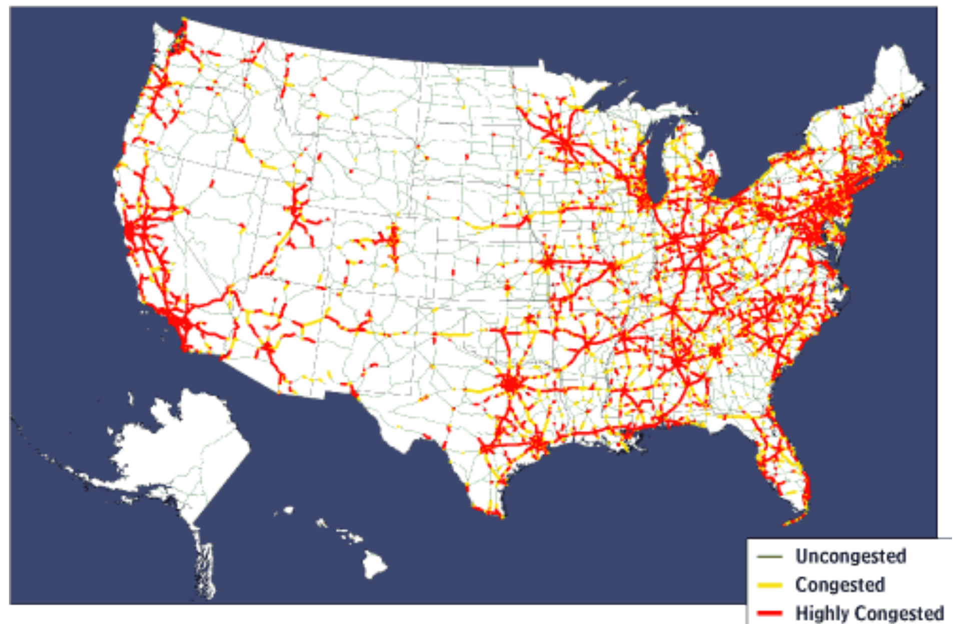

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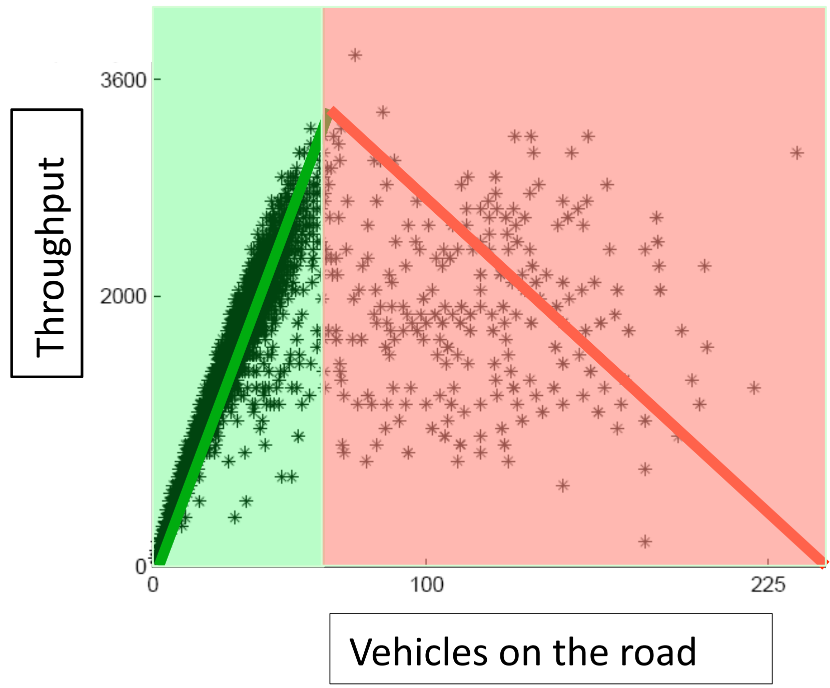
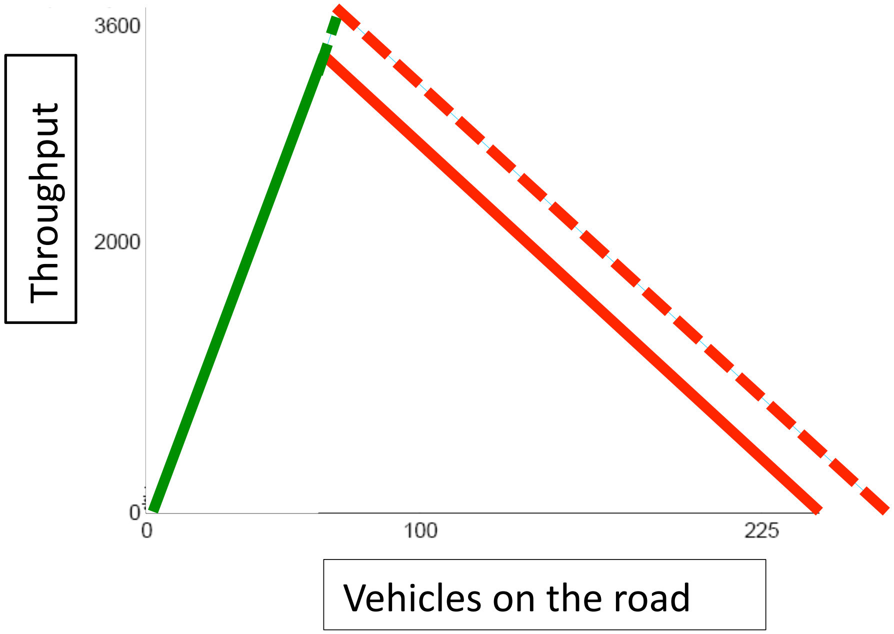
Congestion: road demand greater than road supply



Congestion mitigation strategies

- Capacity increase
  - Roadway expansion
  - Variable speed limits
  - Incident management



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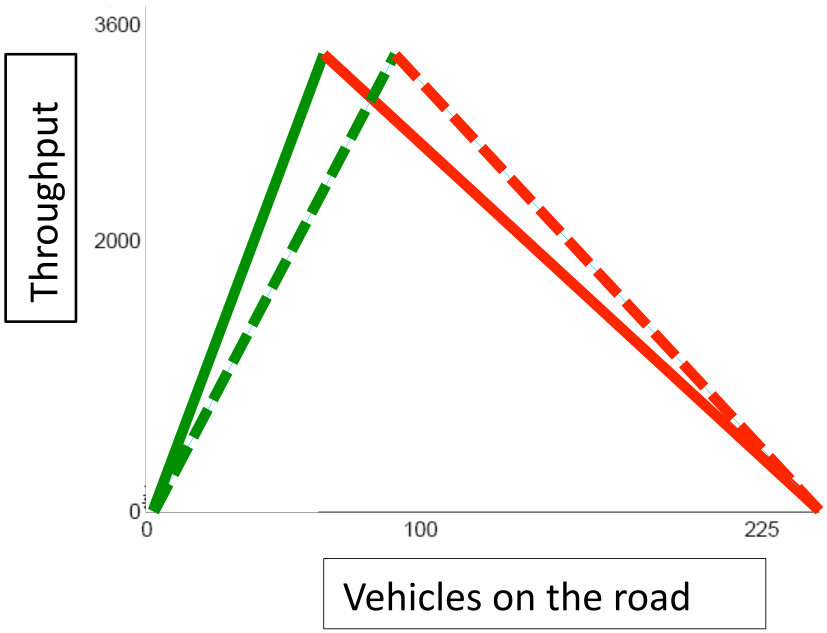
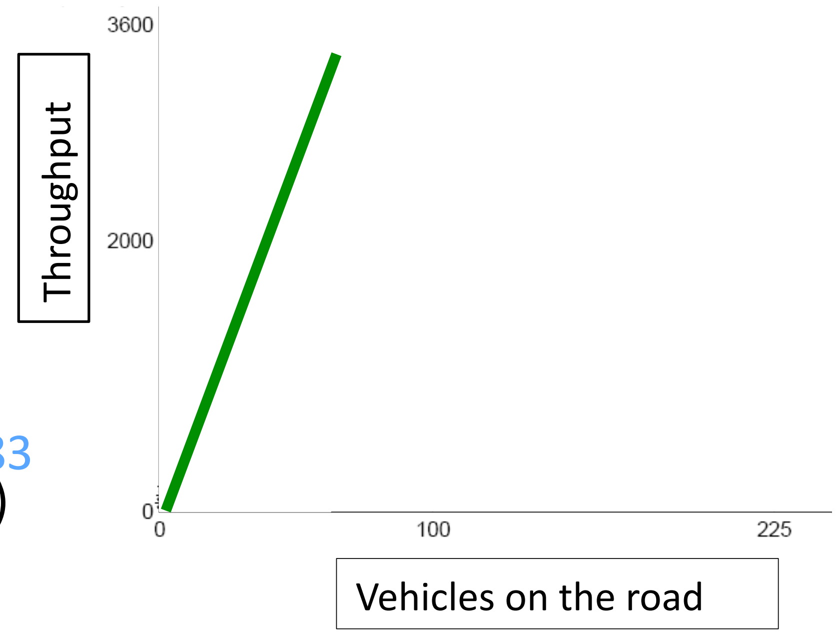


Congestion mitigation strategies

- Demand adjustment
  - Mode shift
    - High-occupancy vehicle lanes
    - Public transportation saved 783 million hours in the US in 2009)
  - Temporal shift
    - Telecommuting
    - Ramp metering: Minnesota(2000), 22% reduction in travel-time
    - Dynamic toll system: Stockholm reduced traffic by 20%, wasted time by 25%)
  - Spatial shift (routing directions)



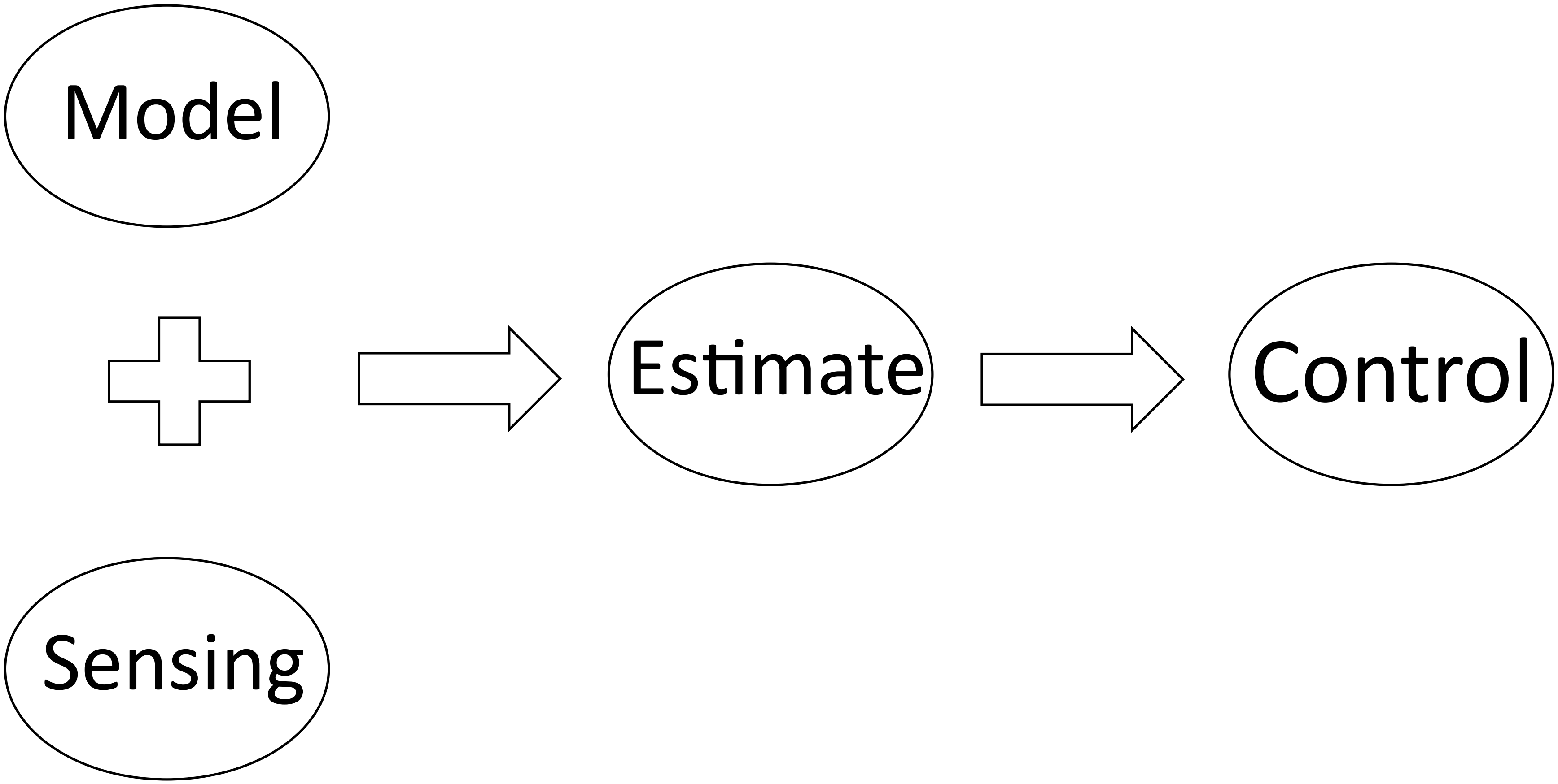
Modeling, estimation, and control

- Traffic modeling
  - Microscopic (vehicular) or macroscopic (elements of flow) perspective
  - Physical principles and statistical assumptions

- Estimation methods
  - Analytical only for specific models and statistics
  - Most computationally-intensive task: tractability for real-time analysis
  - Require assumptions on origin and nature of uncertainty

- Control algorithms
  - Traveler information (congestion maps, routing directions)
  - Traffic assignment (ramp metering, road pricing, variable speed limits)

Modeling, estimation, and control



Classical sensing technologies

loops

video

fleet tracking

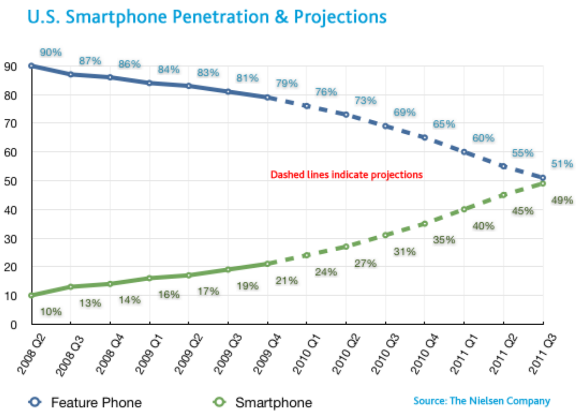
magnetometer

radar

RFID

Smartphone ubiquity: from smart roads to smart drivers

- Spread of mobile and smart phones
  - Worldwide mobile phones market increased b 20% in Q1 of 2011
  - Close to 50% penetration rate in the US

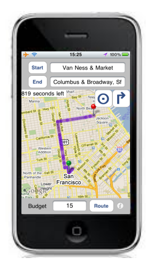


- Dynamic traffic control
  - Accurate real-time information (<5 minutes delay)
  - High-frequency update (>1 per minute)
  - Dynamic routing (Google: March 2011)

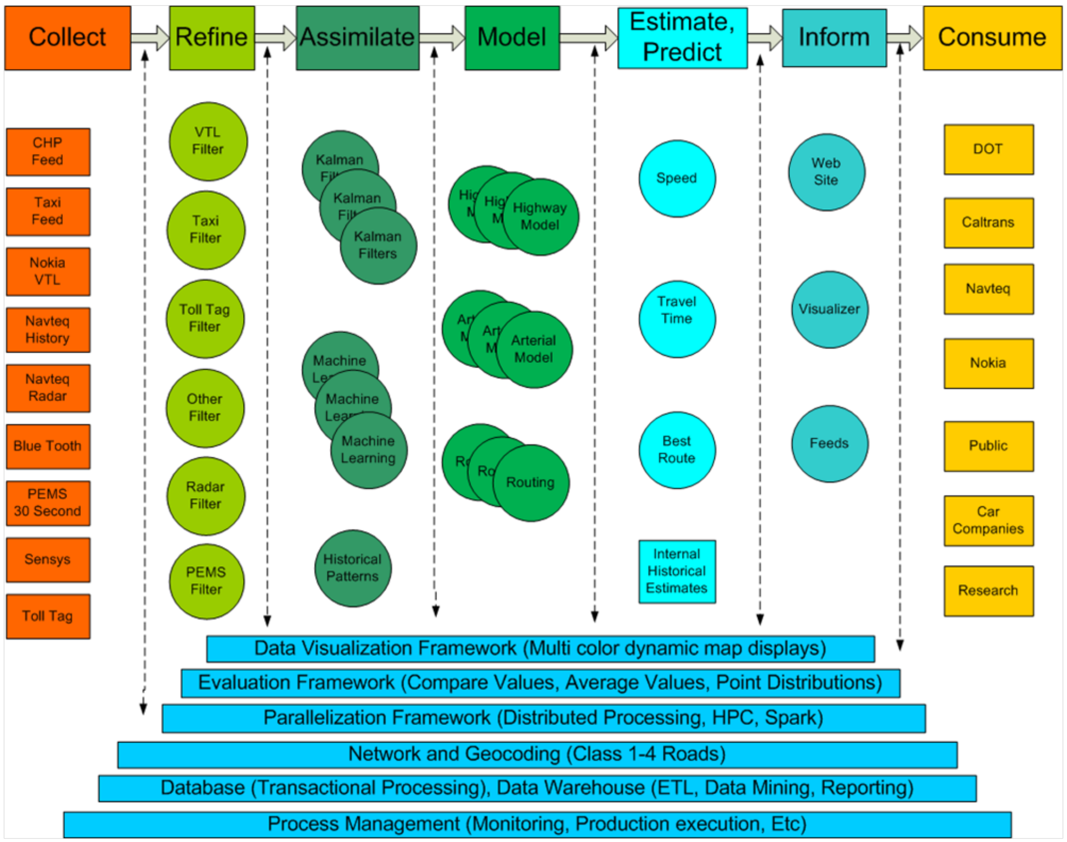
- Adaptive control (appropriate for stochastic systems)
  - Accounts for more complex criteria (reliability)
  - Personalized route recommendations

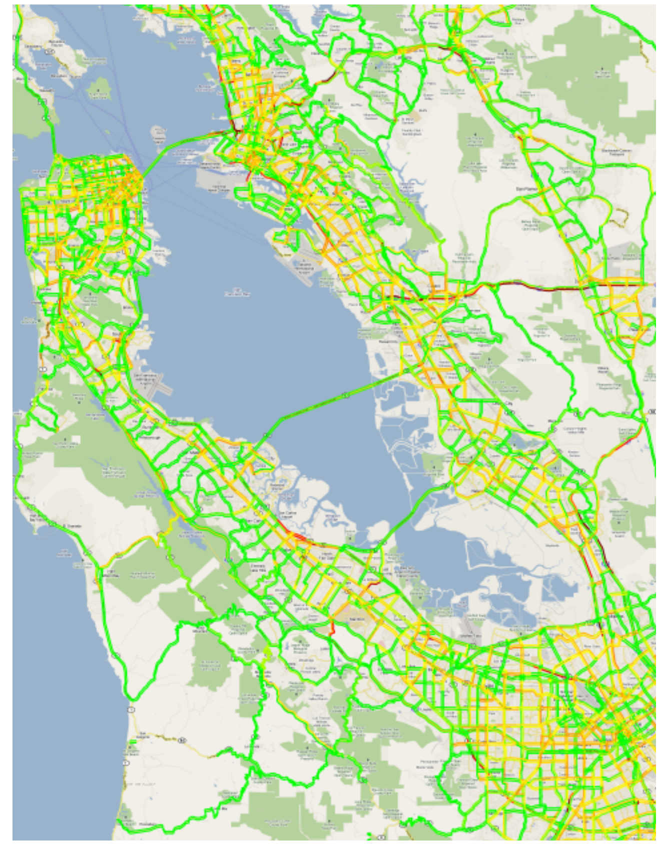


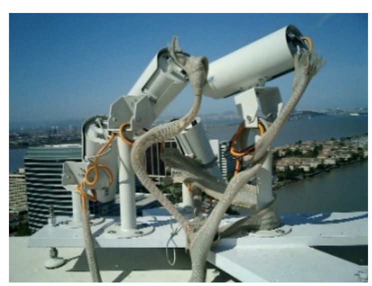


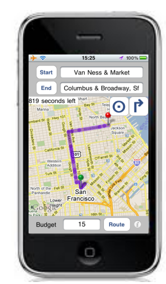


Mobile Millennium: a traffic information system









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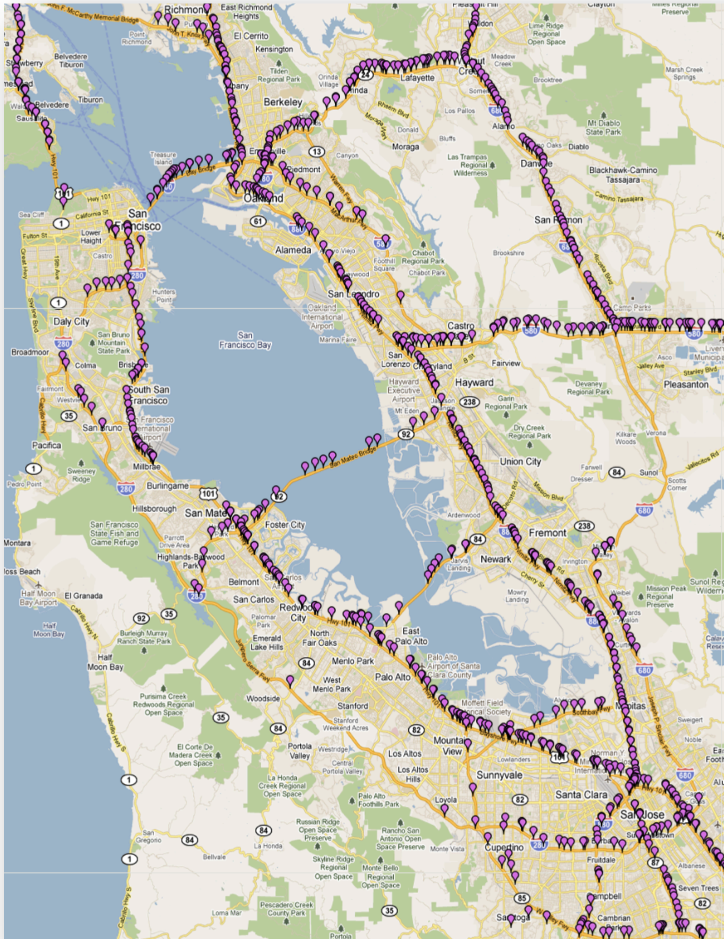
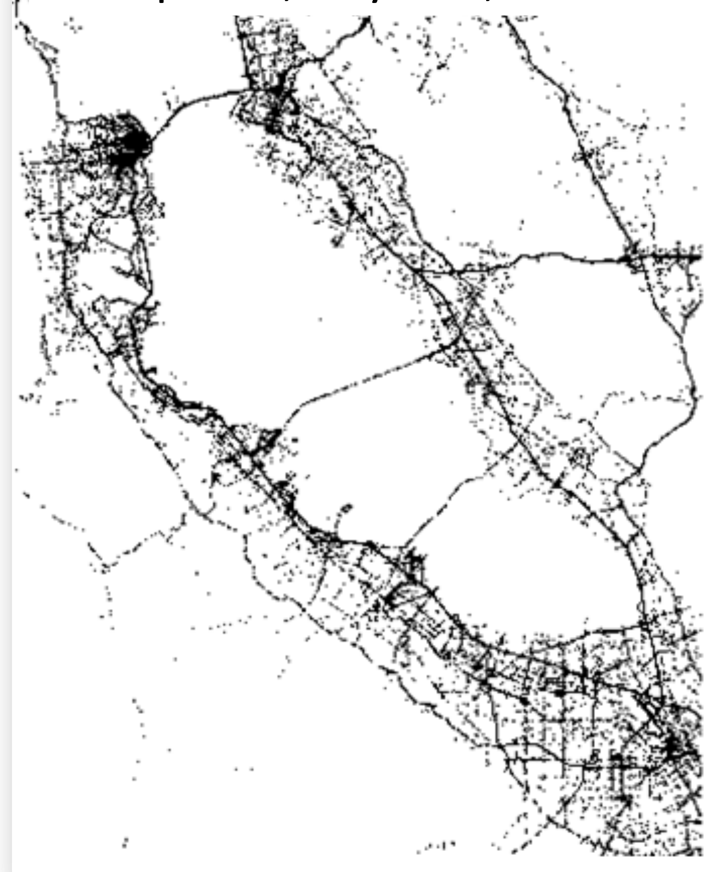
Ubiquitous sensing

PeMS loop detector stations



•Loop detectors
•Count and occupancy
•Localized in space

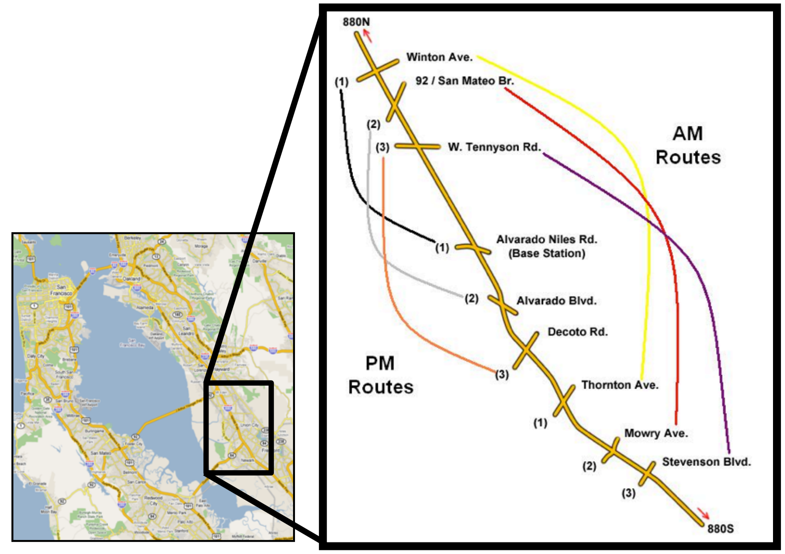
Mobile Millennium, GPS point speeds, July 29th, 2010

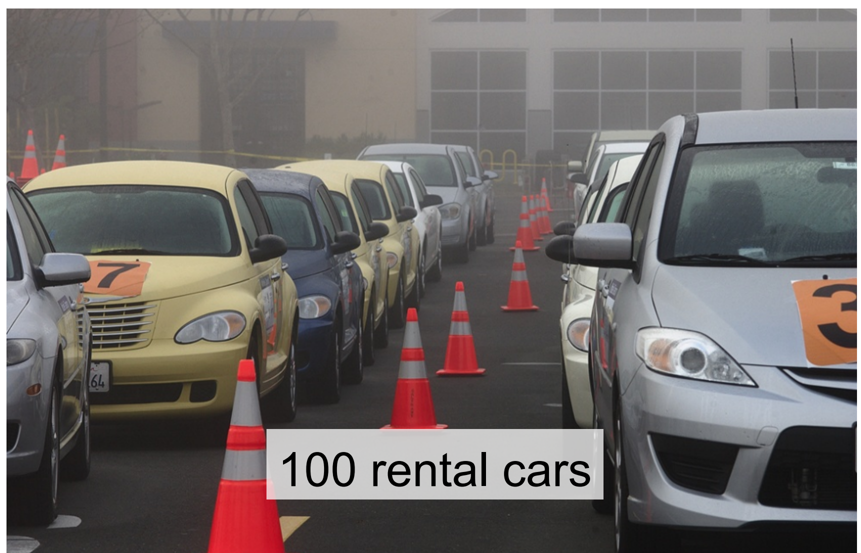


•Personal GPS
•Point speeds
•Distributed across the road network

Mobile Century experiment: proof of concept

- Mobile Century experiment
  - February 8th, 2008
  - 10 miles, 100 cars, 100 GPS-enabled smartphones
  - Accident and morning congestion
  - Proof of concept of added value of GPS data for traffic estimation
  - Collected data available at traffic.berkeley.edu

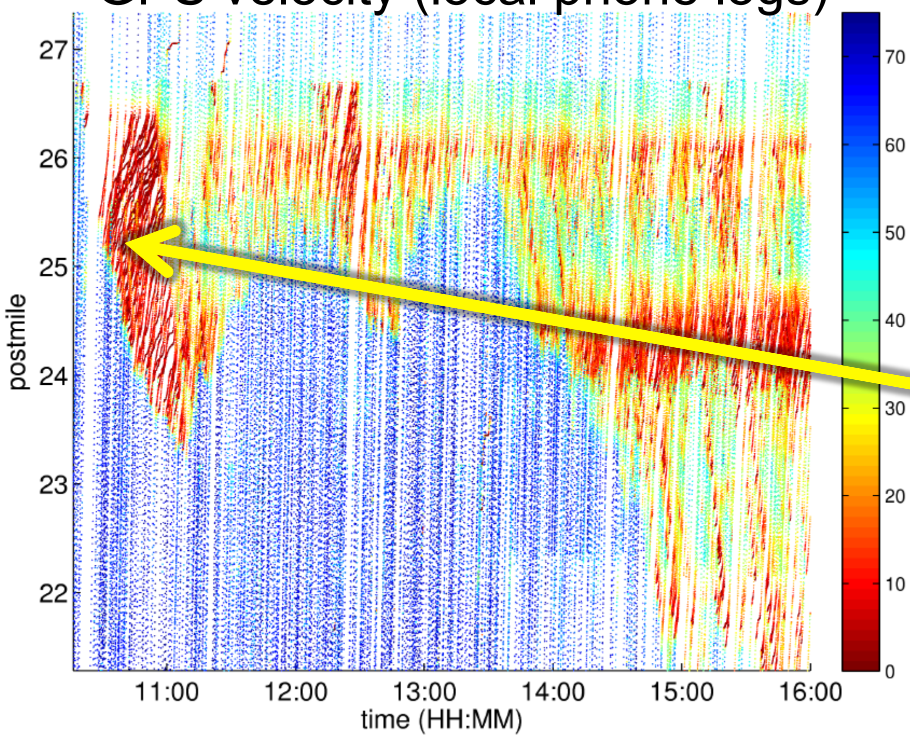




[Herrera, Work, Ban, Herring, Jacobson, Bayen, TR-C, 2010]

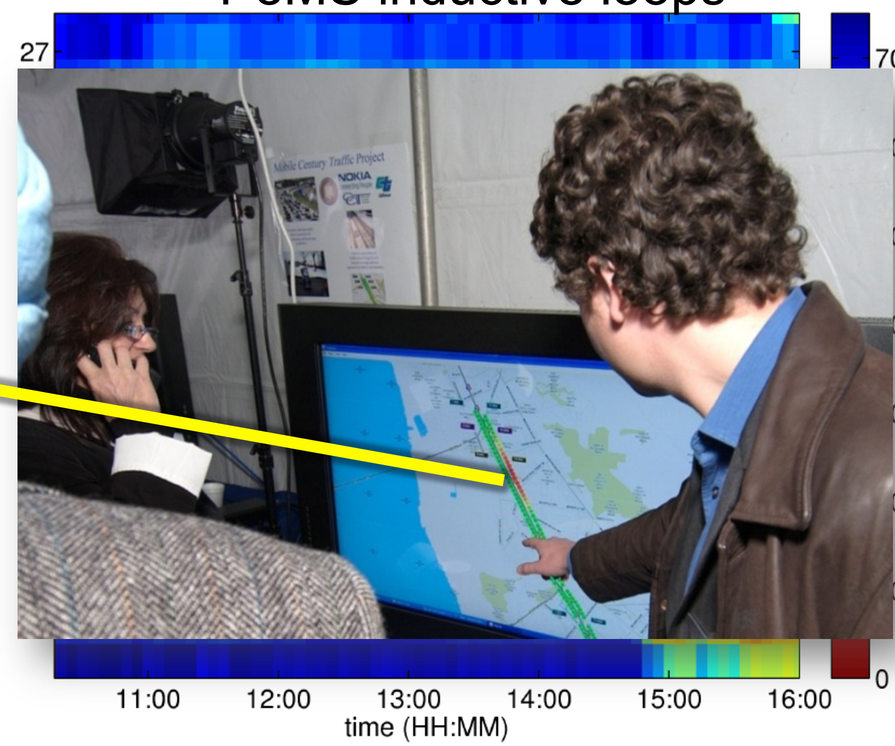
Mobile Century experiment: data collection

GPS velocity (local phone logs)



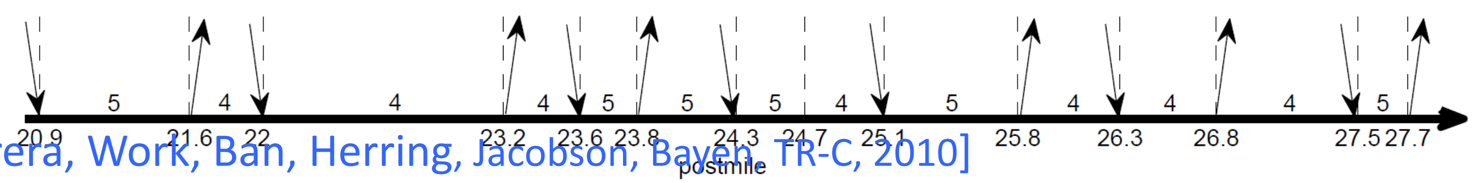
(2-5% of traffic)

PeMS inductive loops



(17 inductive loops)

road geometry: 13 edges, 14 vertices



[Herrera, Work, Ban, Herring, Jacobson, Bayen, TR-C, 2010]

Mobile Century experiment: video data collection

- Video data:
  - Vehicles counts
  - Travel time validation

[Herrera, Work, Ban, Herring, Jacobson, Bayen, TR-C, 2010]

Mobile Century experiment: validation

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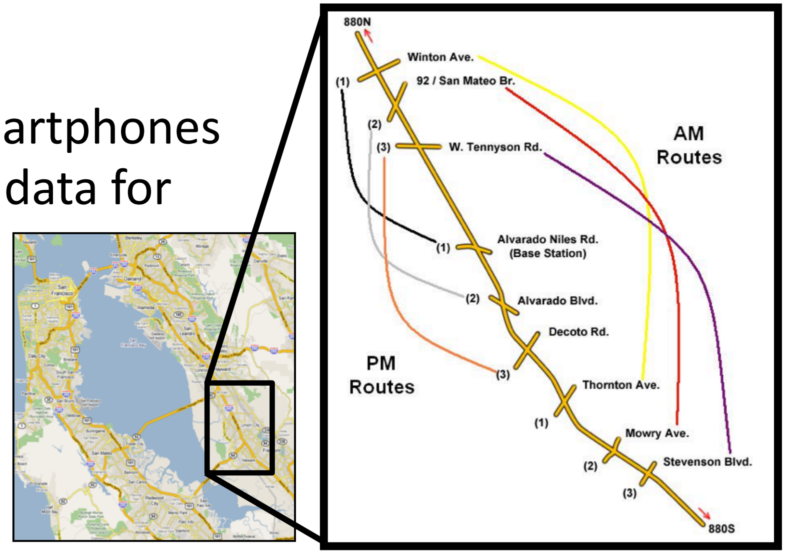

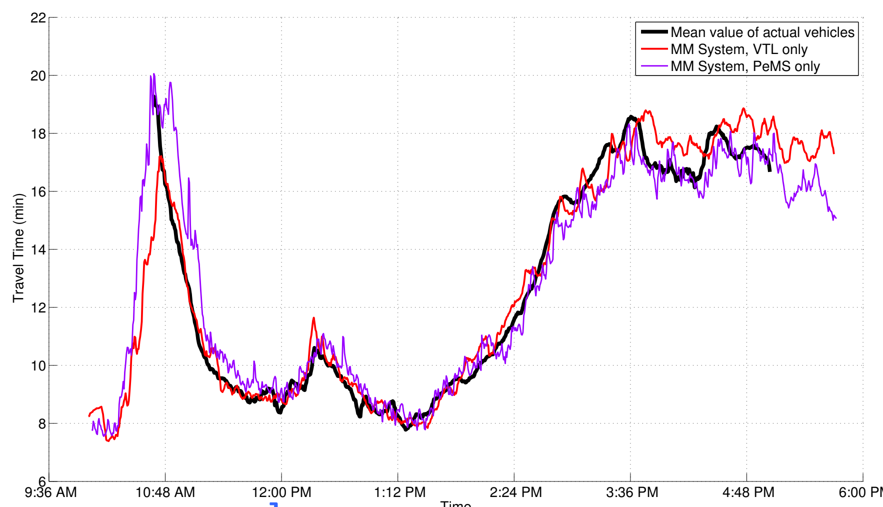

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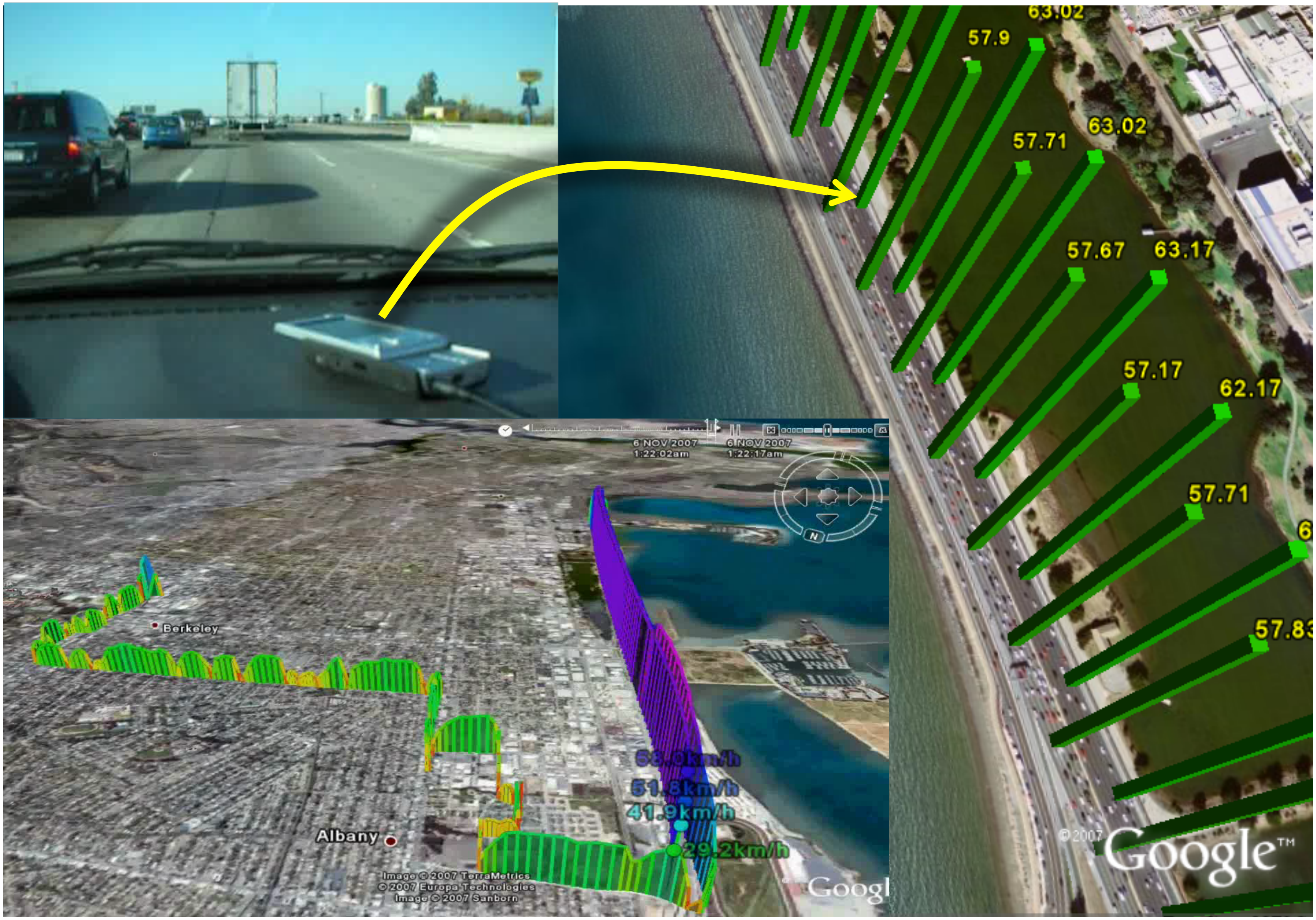
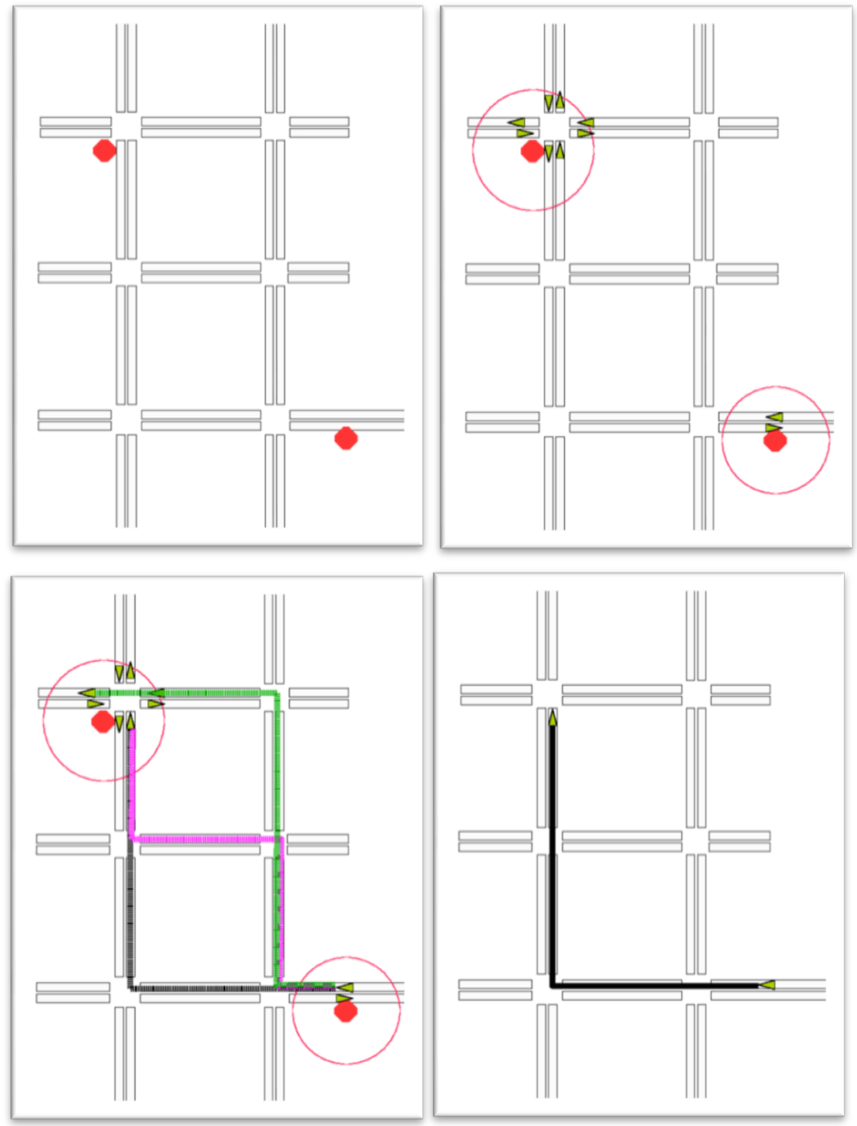
Temporal sampling



Processing temporally sampled data

- Data characteristics
  - Large sampling interval (power and privacy constraints)
  - Noisy measurements (in particular in urban areas)
  - Point data (no speed) must be converted to travel-times
- HMM map-matching algorithm
  - Reduces sensitivity to measurements errors
  - Sensitive to very small speeds



[Hunter, Bayen, TITS, 2011]

Mobile Millennium system: Yellow Cab data

One day of Yellow Cab data: 2010-03-29 04:00:02.0

Mobile Millennium

CALIFORNIA PATH

http://traffic.berkeley.edu

Example of trajectory reconstruction failure

True trajectory

Naïve reconstruction

Raw GPS point

[Hunter, Bayen, TITS, 2011]

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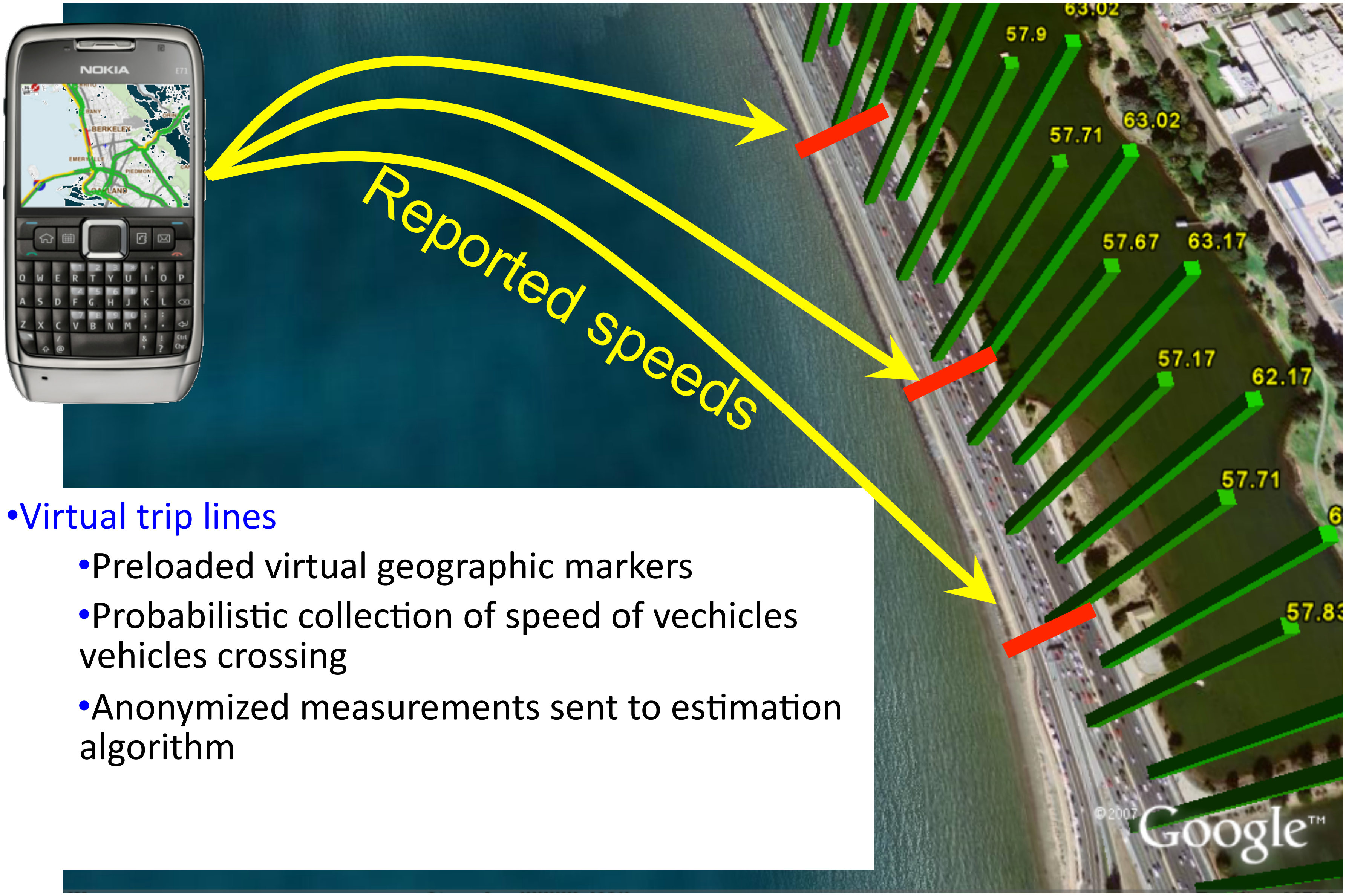
Spatial sampling with Virtual Trip Lines (VTL)



- Virtual trip lines
  - Preloaded virtual geographic markers
  - Probabilistic collection of speed of vechicles vehicles crossing
  - Anonymized measurements sent to estimation algorithm

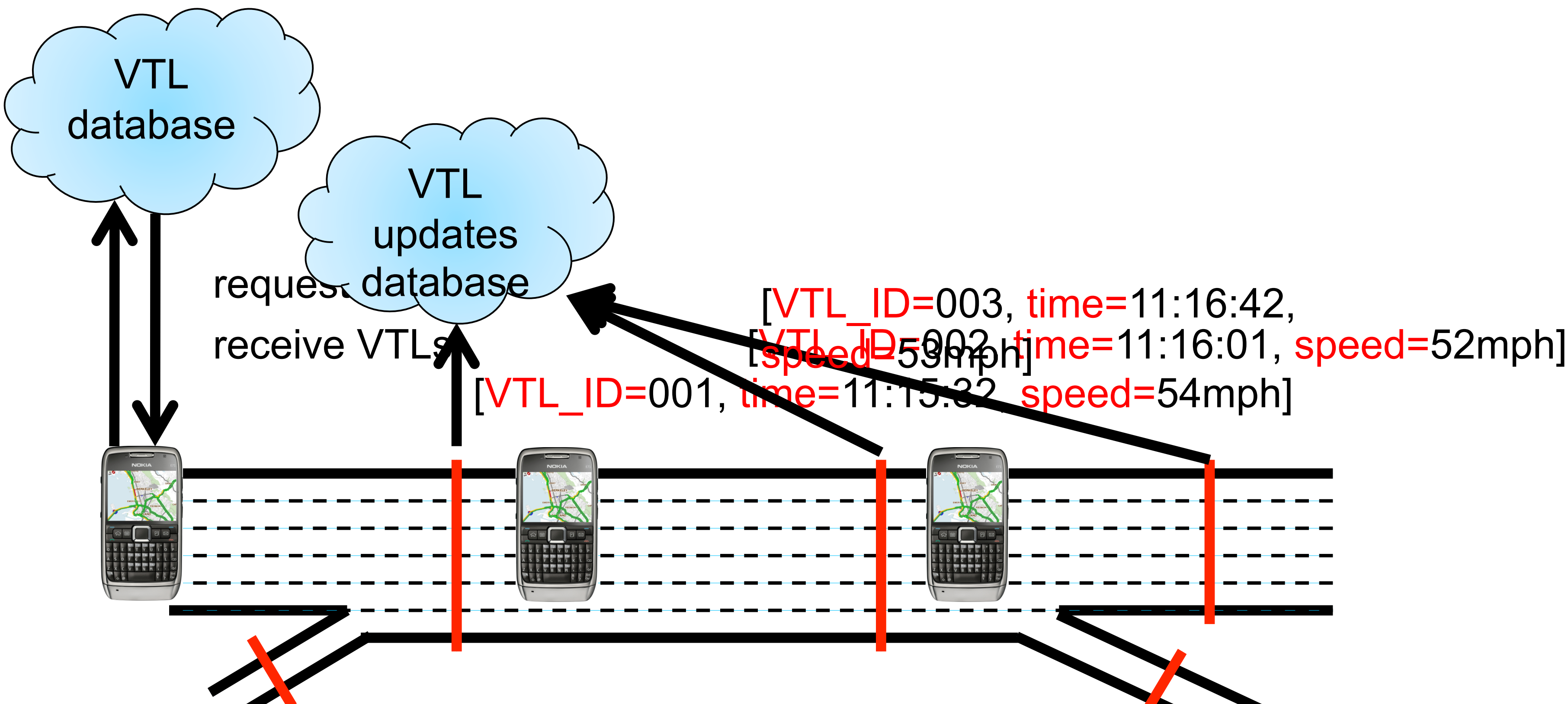
Spatial sampling with Virtual Trip Lines (VTL)

Virtual Trip Line (VTL): virtual trigger to send measurements

step 1: download VTLs to cell phone (automatic)

step 2: check: does my GPS trajectory intersect a VTL?

if yes, send VTL measurement update



[[Hoh, Gruteser, Herring, Ban, Work, Herrera, Bayen, Annavaram, Jacobson, Mobisys 2008]

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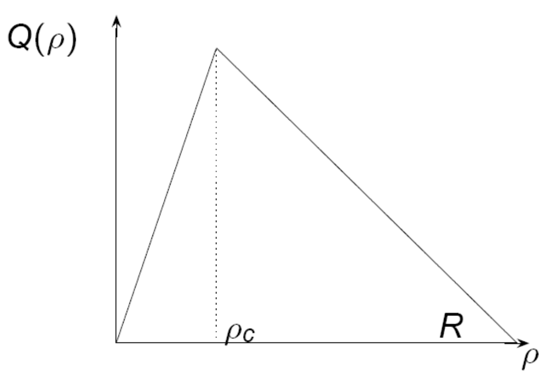
First order scalar conservation law models

- Traffic state: density $\rho(t,x)$ of vehicles at time t and location x
- Dynamics: Scalar one dimensional conservation law, transport equation

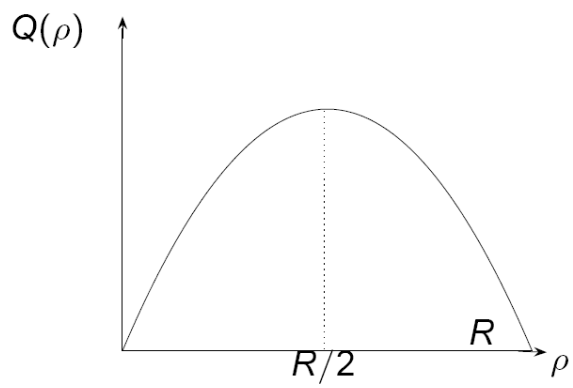
$$\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = 0$$

- Empirical flux function: the fundamental diagram

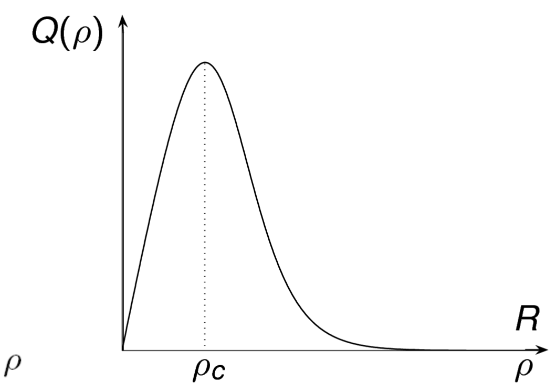
Newell-Daganzo



Greenshields



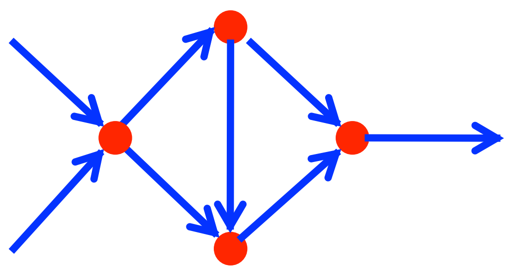
Kerner, Papageorgiou,Li



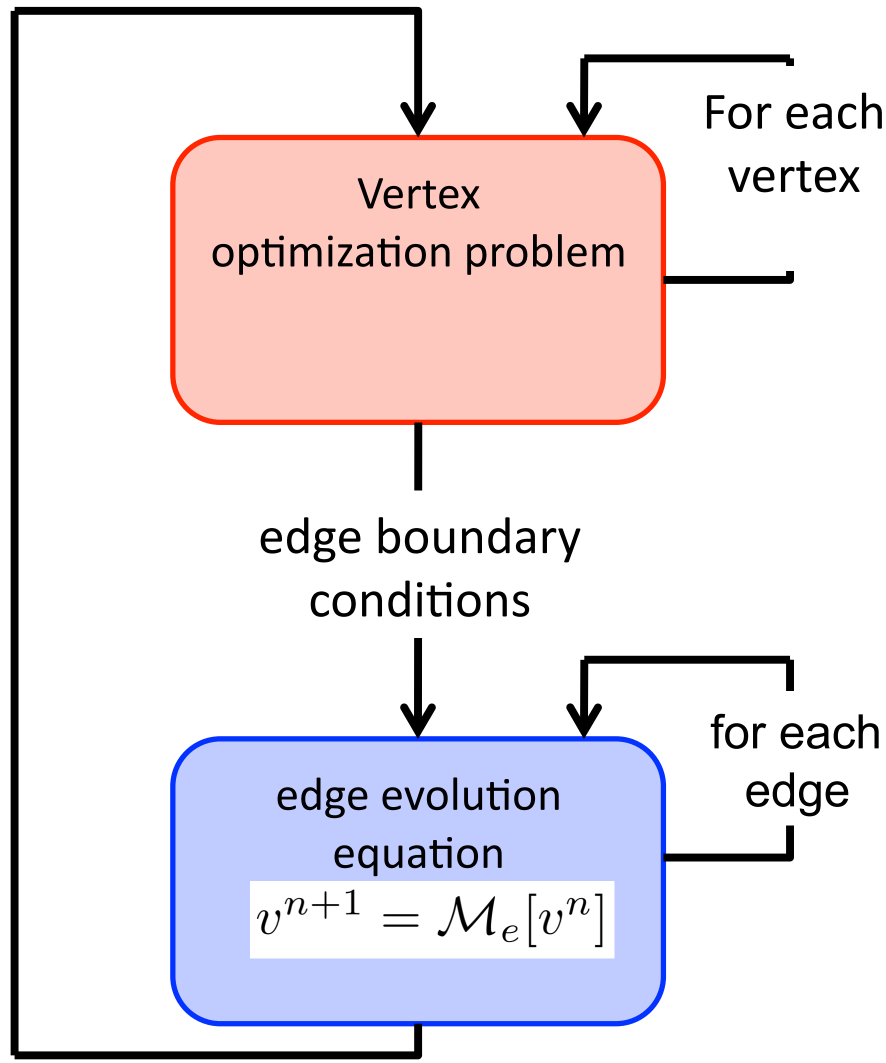
[Lighthill, Whitham, 1955], [Richards, 1956], [Greenshields, 1935]

Network conservation law model for velocity

- road network as directed graph:
  - → edges
  - ● vertices



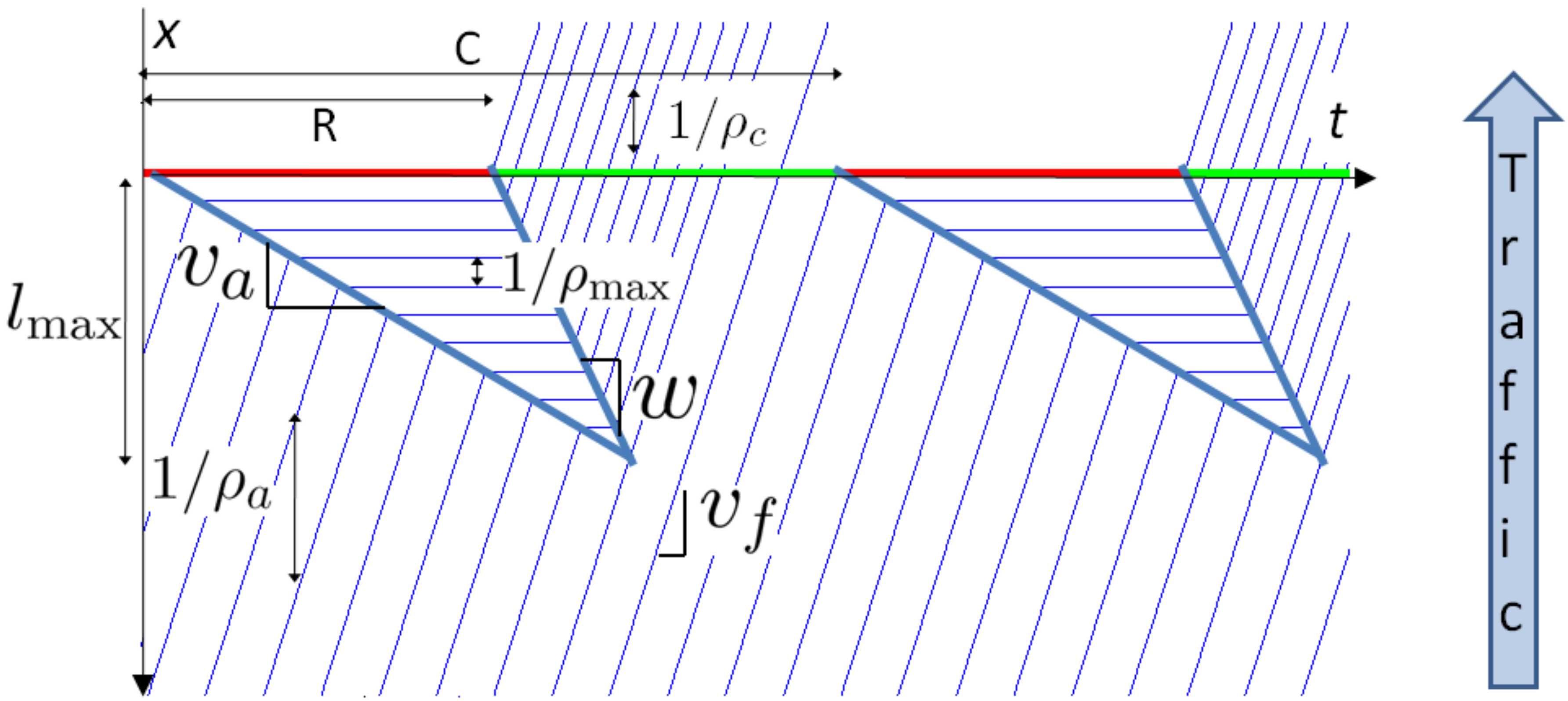
- vertex linear program
  - solves for link boundary conditions
  - mass conserving
  - guarantees uniqueness of solution on networks



[Work, Blandin, Tossavainen, Piccoli, Bayen, AMRX, 2010]

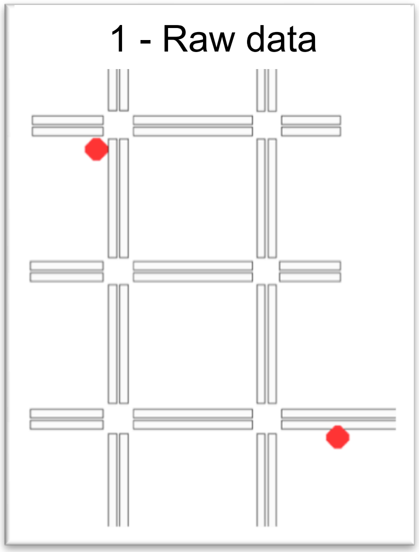
Flow-based statistical model of urban traffic

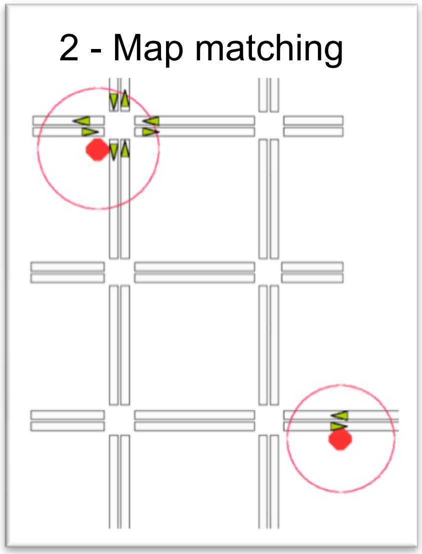
- Macroscopic model at traffic lights
- Queue propagation implies specific spatial distribution of vehicles
- Estimated travel-time distributions have to be consistent with queue phenomena

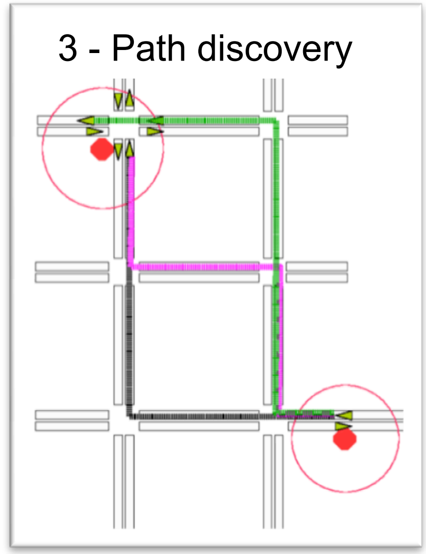


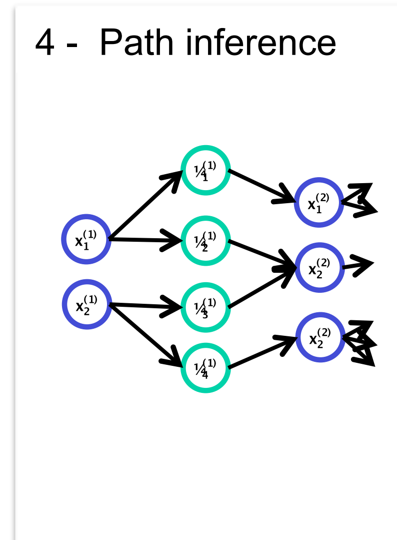
[Hofleitner, Herring, Bayen, TR-B, 2011]

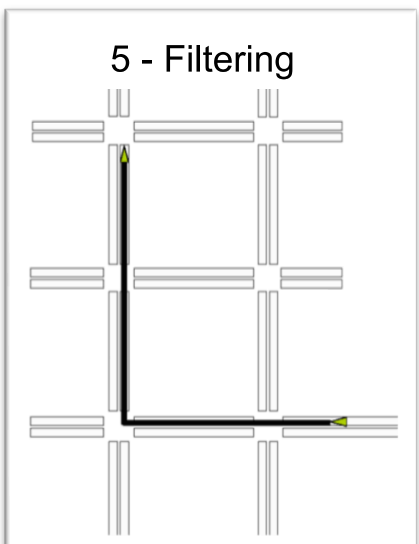
Machine learning model of travel-time

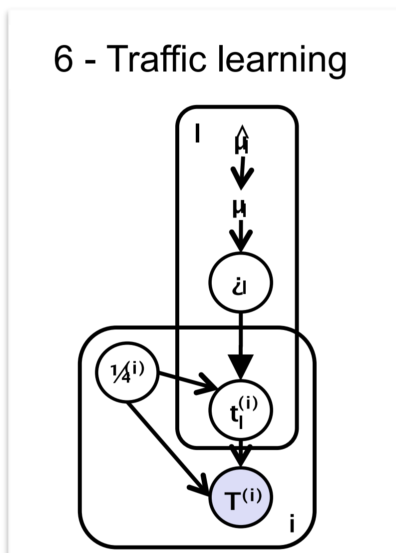


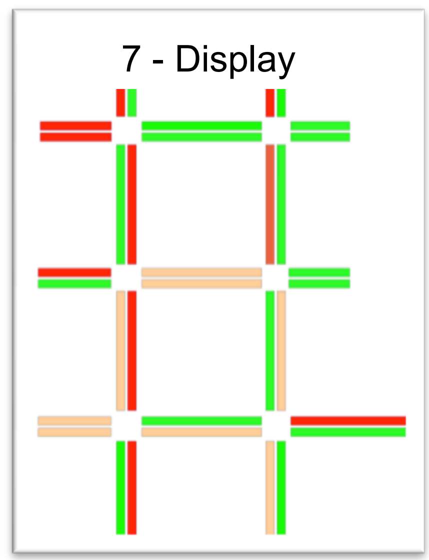












[Hunter, Bayen, TITS, 2011]

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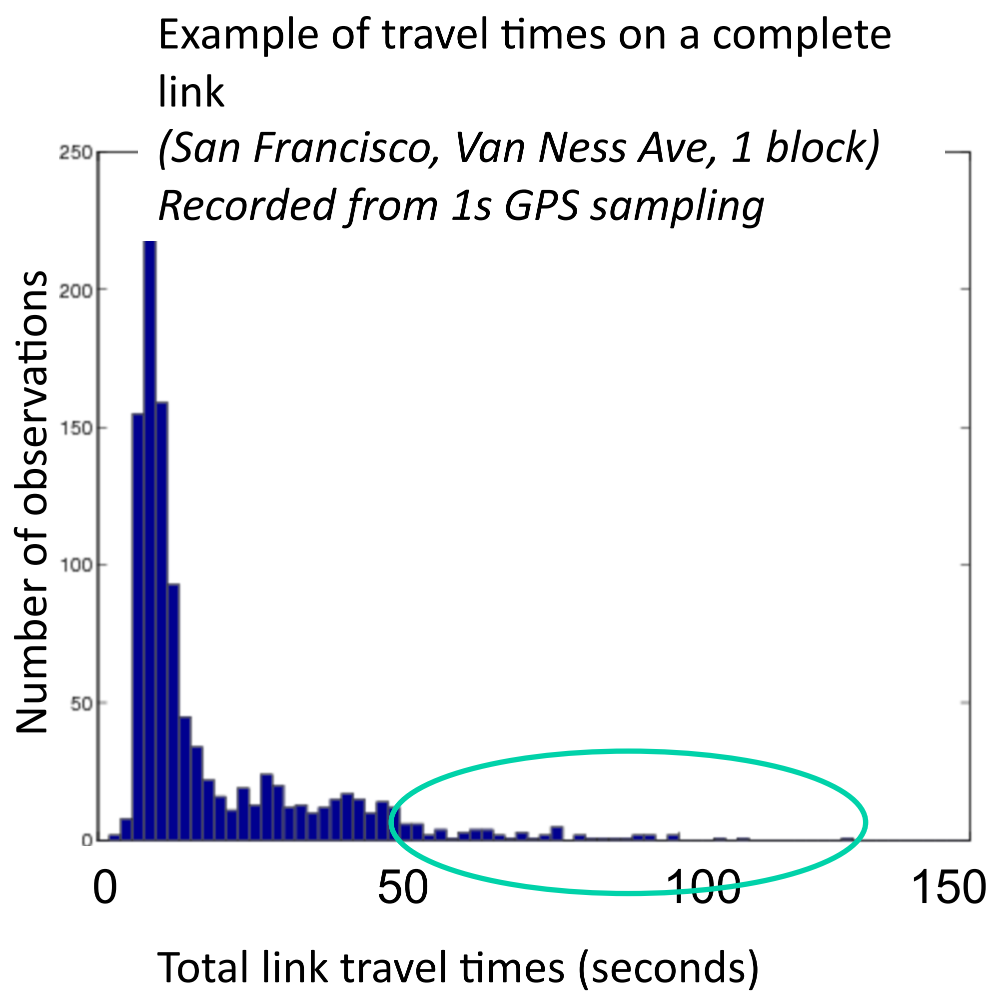
Properties of urban travel times

- Non gaussian distributions
- Link independence do not hold

- Required distributions
  - Heavy tailed
  - Mixture model



[Hunter, Bayen, TITS, 2011]

Properties of traffic flow

- Nonlinearities of traffic flow
  - Globally: development of congestion phases
  - Locally: instabilities leading to stop-and-go waves

- Nonlinearities
  - Induce discontinuities in the solution of the PDE
  - Have to be accounted for as mixture distributions

- Nondifferentiability
  - Caused by the existence of stationary shockwaves
  - Break assumptions of Taylor series approximations

[Blandin, Couque, Bayen, Work, TR-B, 2011]

The ensemble Kalman filter

Initialization: Draw *K* ensemble realizations $v_a^0(k)$ with $k \in \{1, \cdots, K\}$ from a process with a mean speed $\bar{v}_a^0$ and covariance $\mathbf{P}_a^0$.

Forecast: Update each of the *K* ensemble members according to the discrete velocity model. Then update the ensemble mean and covariance according to:

$$v_f^n(k) = \mathcal{M}[v_a^{n-1}(k)] + \eta^n(k)$$

$$\bar{v}_f^n = \frac{1}{K}\sum_{k=1}^{K} v_f^n(k)$$

$$\mathbf{P}_{\mathrm{ens},f}^n = \frac{1}{K-1}\sum_{k=1}^{K} \left(v_f^n(k) - \bar{v}_f^n\right)\left(v_f^n(k) - \bar{v}_f^n\right)^T$$

Analysis: Obtain measurements, compute the Kalman gain, and update the network forecast:

$$\mathbf{G}_{\mathrm{ens}}^n = \mathbf{P}_{\mathrm{ens},f}^n \left(\mathbf{H}^n\right)^T \left(\mathbf{H}^n \mathbf{P}_{\mathrm{ens},f}^n \left(\mathbf{H}^n\right)^T + \mathbf{R}^n\right)^{-1}$$

$$v_a^n(k) = v_f^n(k) + \mathbf{G}_{\mathrm{ens}}^n \left(y_{\mathrm{meas}}^n - \mathbf{H}^n v_f^n(k) + \chi^n(k)\right)$$

[Work, Blandin, Tossavainen, Piccoli, Bayen, AMRX, 2010]

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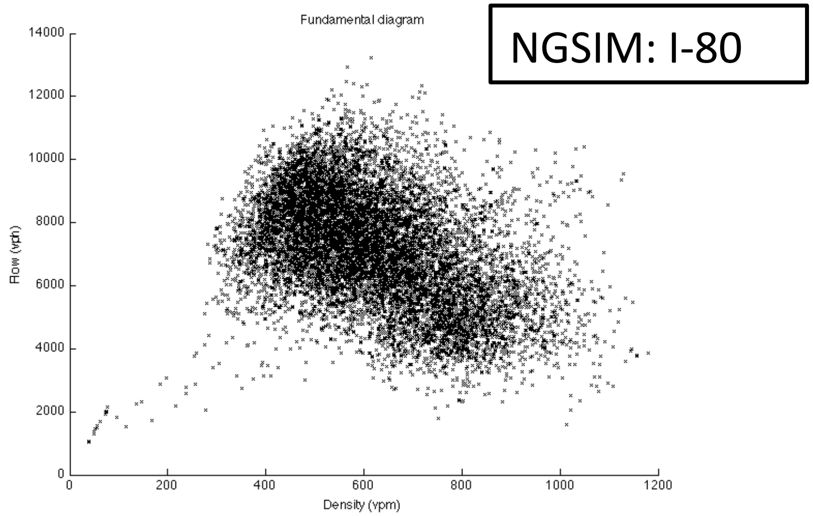
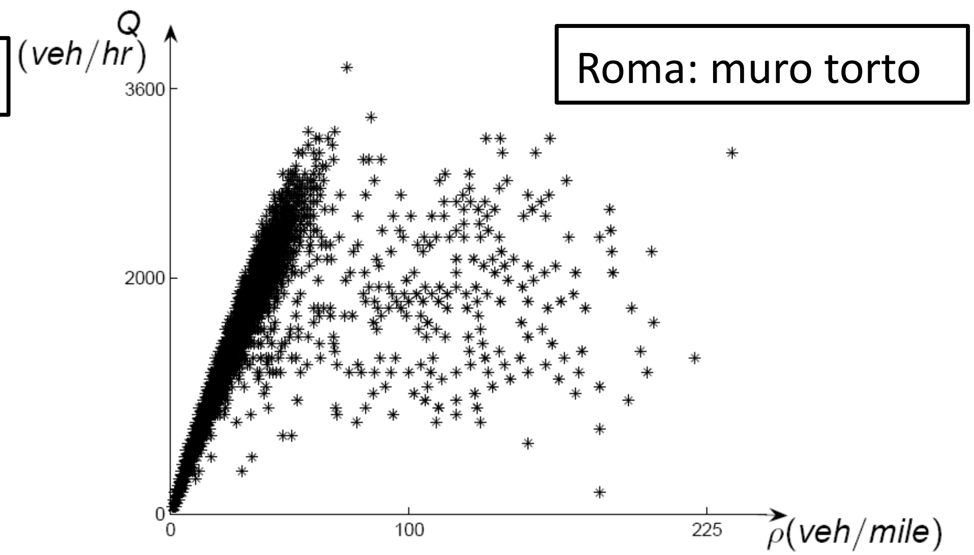
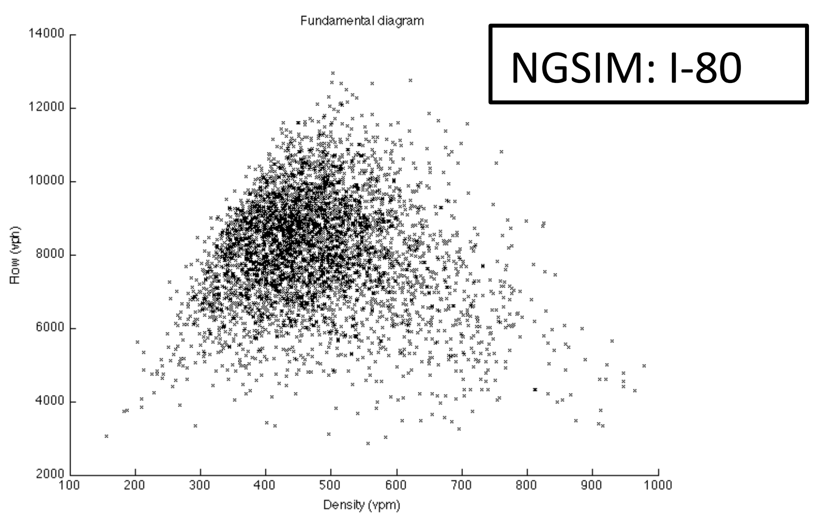
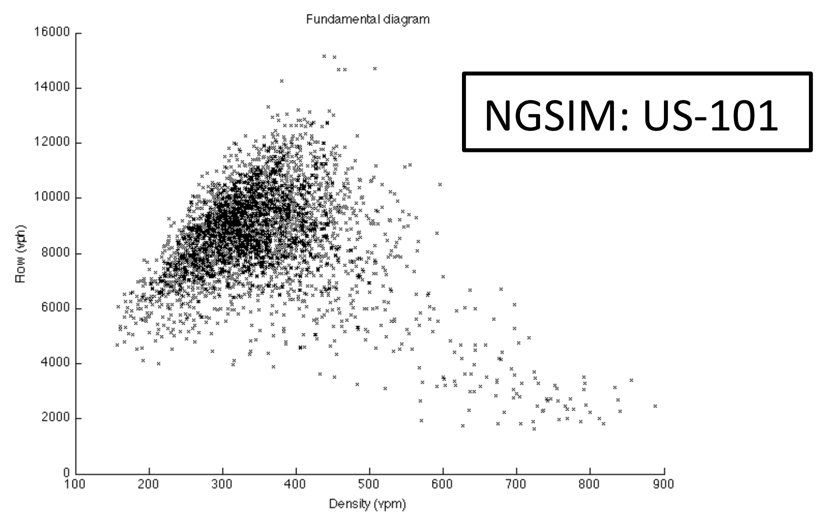
Behavioral fluid mechanics





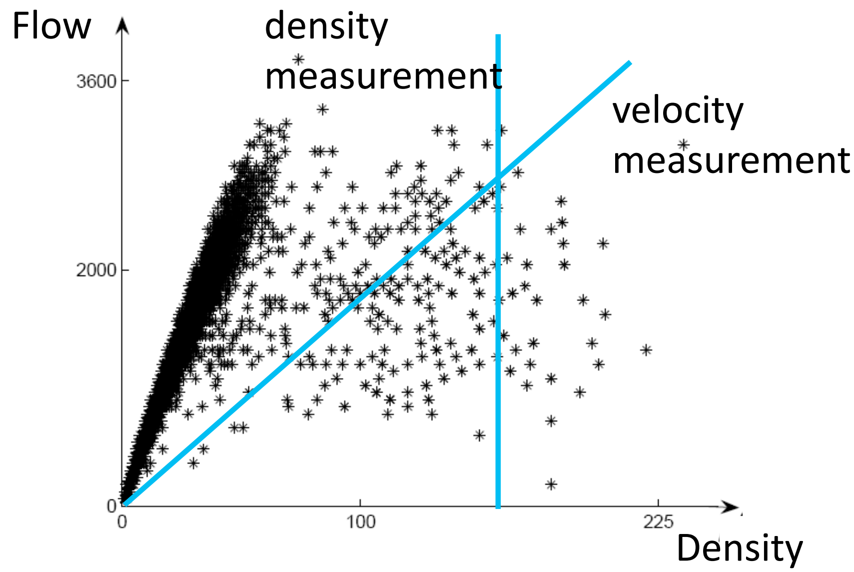


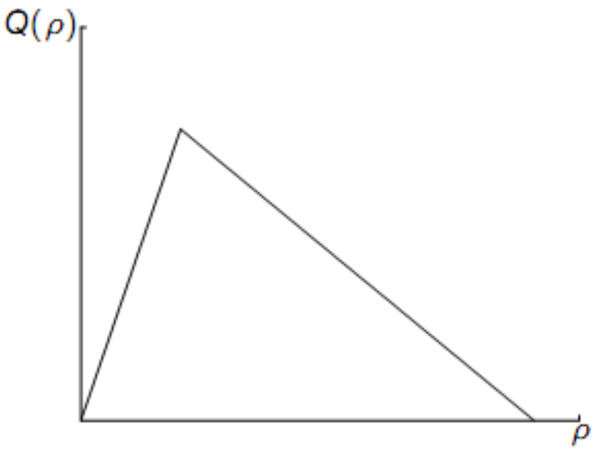


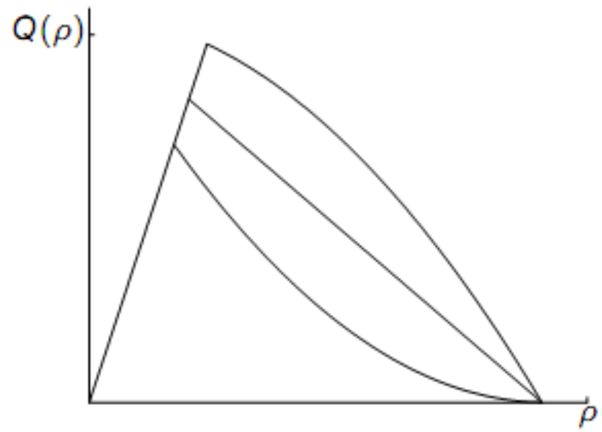
[Blandin, Bretti, Cutolo, Piccoli, AMC, 2009]

Macroscopic models of set-valued driving behavior

- Extension of classical conservation law framework
- Modeling choice of different speeds for different densities
- Integration of joint velocity and density measurements







[Blandin, Work, Goatin, Piccoli, Bayen, SIAP, 2011]

Macroscopic models of set-valued driving behavior

- Definition of traffic state as
  - ρ in free-flow phase
  - (ρ, q) in congestion phase
- Definition of the standard speed function

$$v^{\mathrm{s}}(\rho) = \begin{cases} V & \text{in free-flow} \\ v^{\mathrm{s}}_{\mathrm{c}}(\rho) & \text{in congestion} \end{cases}$$

where $v^{\mathrm{s}}_{\mathrm{c}}(\cdot)$ is smooth with positive values.

- Definition of speed as a perturbation around the standard speed in congestion

$$\boxed{v = \begin{cases} V & \text{in free-flow} \\ v_{\mathrm{c}}(\rho, q) := v^{\mathrm{s}}_{\mathrm{c}}(\rho)(1+q) & \text{in congestion} \end{cases}}$$

- Definition of two distinct dynamics

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, v) = 0 & \text{in free-flow} \\ \begin{cases} \partial_t \rho + \partial_x(\rho\, v) = 0 \\ \partial_t q + \partial_x(q\, v) = 0 \end{cases} & \text{in congestion} \end{cases}$$

[Blandin, Work, Goatin, Piccoli, Bayen, SIAP, 2011]

Outline

1. Traffic sensing: from eulerian sensors to lagrangian sensors
    1. Crowdsourced probe measurements
    2. Temporal sampling and map-matching
    3. Spatial sampling: virtual trip lines

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    1. Statistical and physical models
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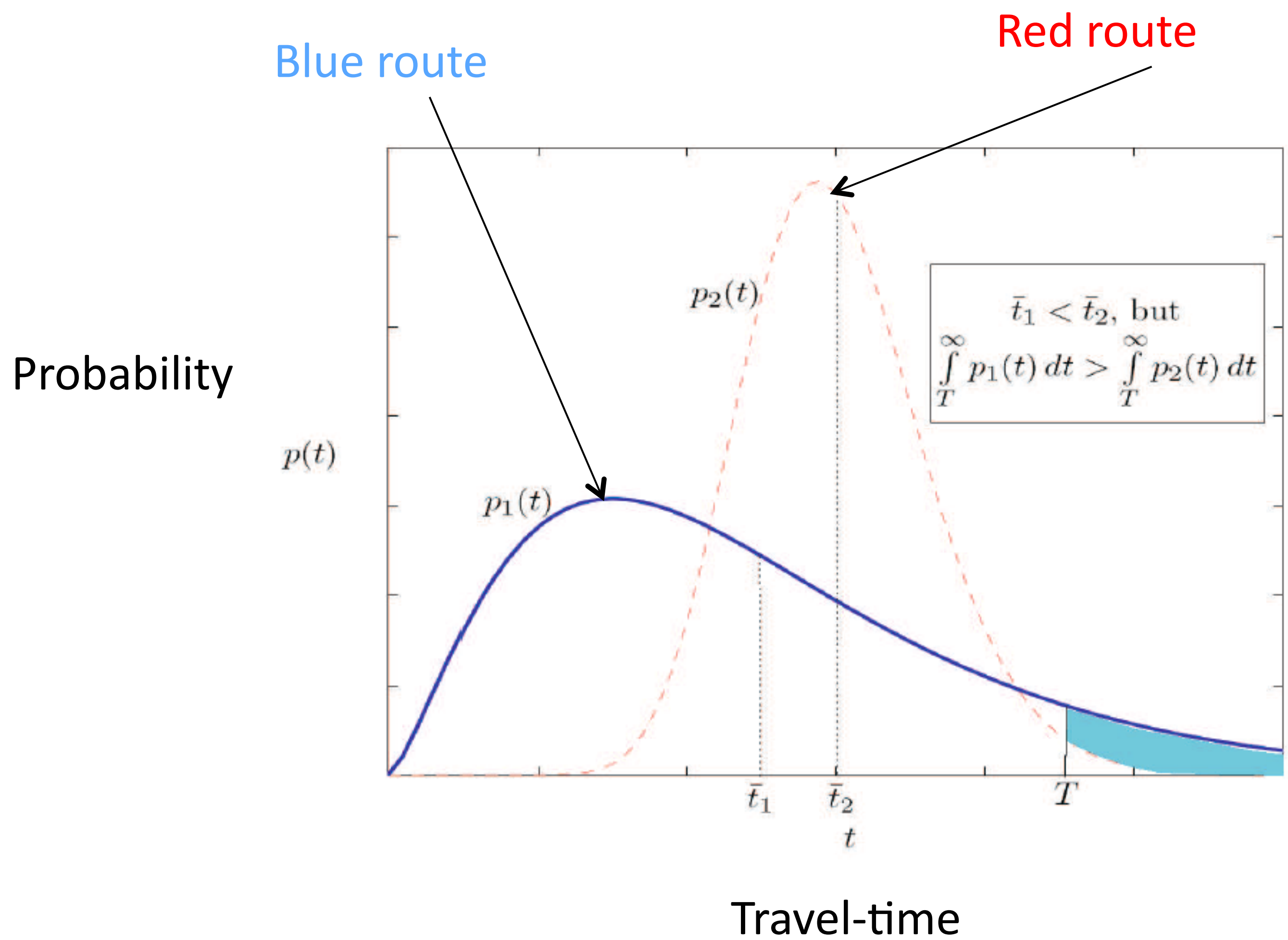
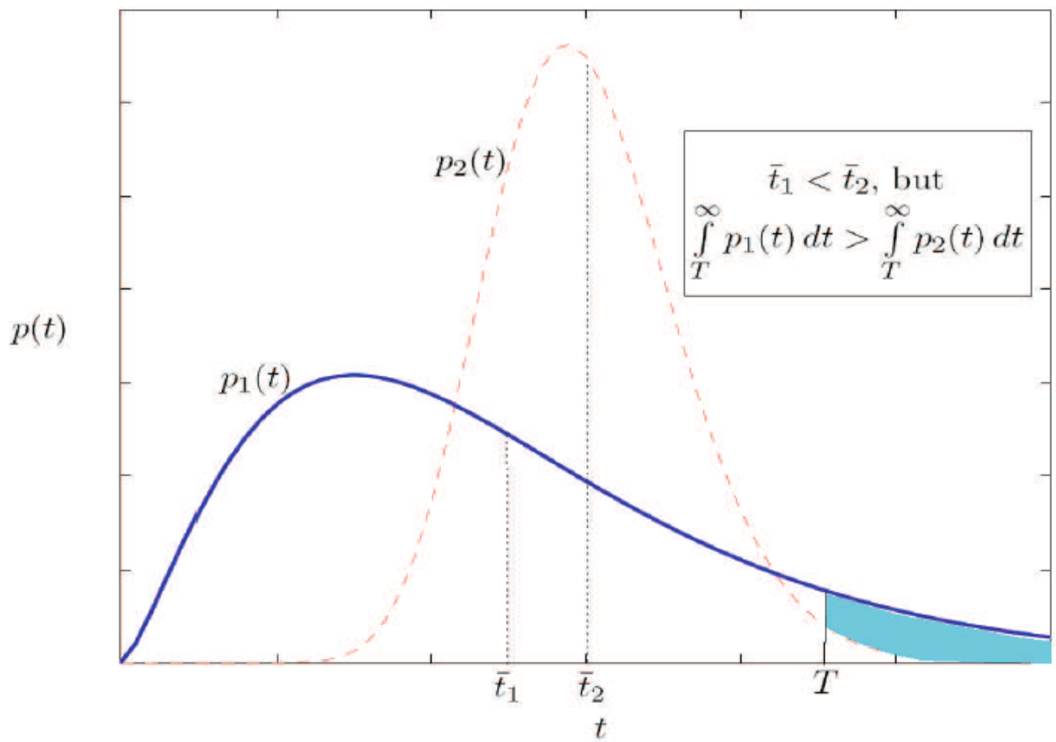
Travel-time reliability



[Loui, 1983]

A stochastic routing formulation: on-time arrival

- **Definition**: Given an origin, a destination, a time budget, maximize the probability of arriving at the destination within budget.



- Challenges:
  - Proof of theoretical convergence
  - Tractable solution algorithm

[Loui, 1983]

Problem statement

- Let $\mathcal{G}(V, E)$ be a directed graph which represents the road network.

- With each edge $e_{ij} \in E$ is associated a continuous link travel time probability density function $p_{ij}(\cdot)$.

- Stochastic on-time arrival (SOTA) problem: given a couple origin-destination $s - t$ and a time budget T, find the optimal routing policy

$$u_i(\tau) := \max_j \int_0^\tau p_{ij}(w)\, u_j(\tau - w)\, dw$$
$$\forall i \in V, i \neq t, (i,j) \in E, 0 \leq \tau \leq T$$
$$u_t(\tau) = 1 \quad 0 \leq \tau \leq T$$

- Challenges
  - Optimal policy may contain loop, no bound on finite number of iterations (value iteration, Picard method of successive approximations)
  - Numerical tractability for real-time applications

[Fan, Kalaba, Moore, 2005]

Algorithm

- Note δ the minimal link travel-time
    - Bound on number of iterations in continuous setting $L = \lceil T/\delta \rceil$

- The convolution product is computed
    - Using the fast fourier transform
    - By increments of size δ

- Complexity $O\left(m \sum_{k=1}^{T/\delta} \frac{k\,\delta}{\Delta t} \log \frac{k\,\delta}{\Delta t}\right)$

compared to brute force method:

$$O\left(m \sum_{k=1}^{T/\Delta t} k\right)$$

Step 0. Initialization.

$u_i^0(t) = 0, \ \ \forall i \in N, \ \ i \neq s, \ \ t \in (0, T)$

$u_s^0(t) = 1, \ \ \forall\, t \in (0, T)$

Step 1. Update

For $k = 1, 2, \ldots, L$

$\tau^k = k\delta$

$u_i^k(t) = u_i^{k-1}(t)$

$\forall i \in N, \ \ i \neq s, \ \ t \in (0, \tau^k - \delta]$

$u_i^k(t) = \max_j \int_0^t p_{ij}(\omega)\, u_j^{k-1}(t - \omega)\, d\omega$

$\forall i \in N, \ \ i \neq s, \ \ (i, j) \in A, \ \ t \in (\tau^k - \delta, \tau^k]$

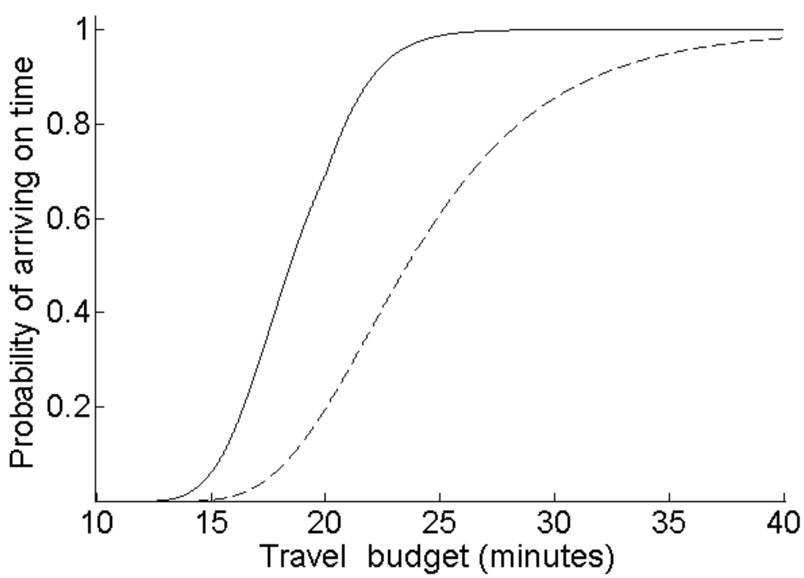
- Improvements relying on local values of minimal link travel-time
    - Definition of policy update invariant $\tau_i \leq \min_j(\tau_j + \delta_{ij})$ and optimal order

[Samaranayake, Blandin, Bayen, ISTTT, 2011]

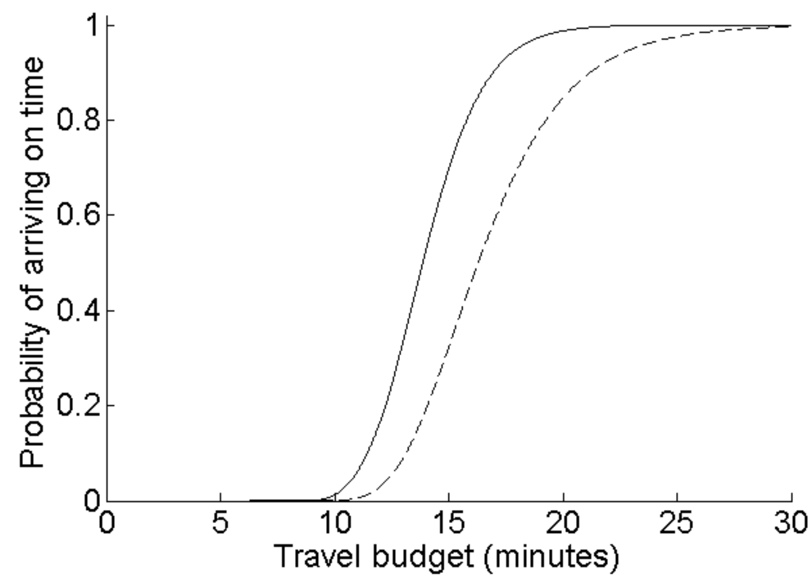
Probability of arriving on time

•San Francisco arterial network

Long route (SOTA policy vs LET path)



Short route (SOTA policy vs LET path)

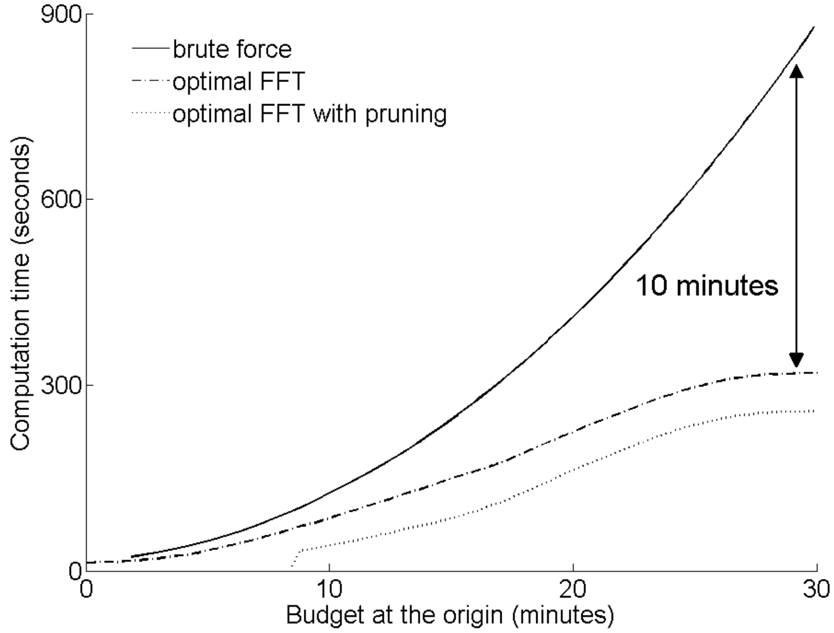


•Dashed: least-expected travel time path
•Solid: stochastic on-time arrival policy

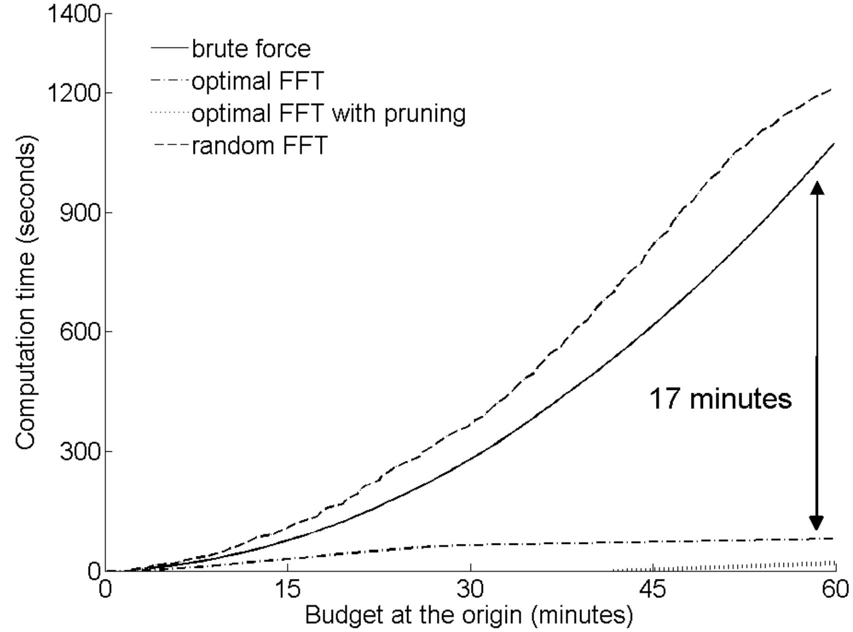
[Samaranayake, Blandin, Bayen, ISTTT, 2011]

Runtime improvement

San Francisco arterial network



Bay Area Highway network



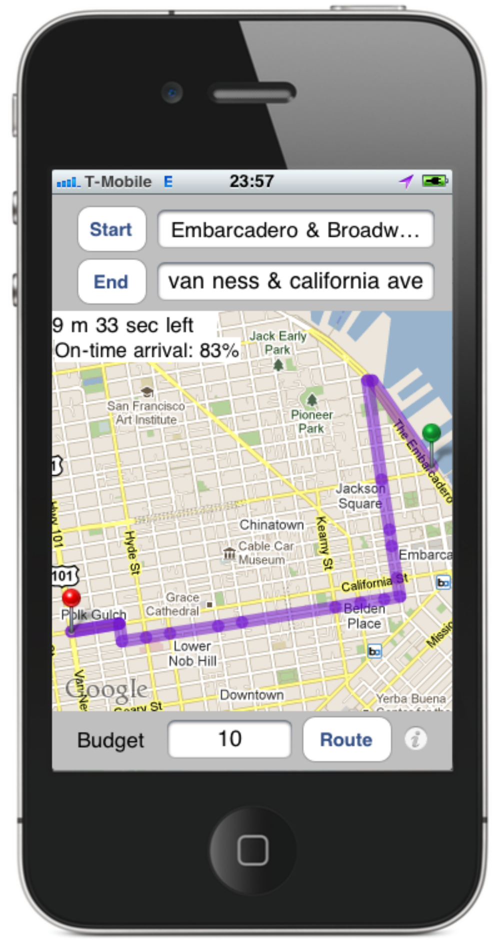
•The algorithm is coded in Java and executed on a Windows 7 PC with a 2.67Ghz Dual Core Intel Itanium processor and 4GB of RAM

[Samaranayake, Blandin, Bayen, ISTTT, 2011]

SOTA iPhone app

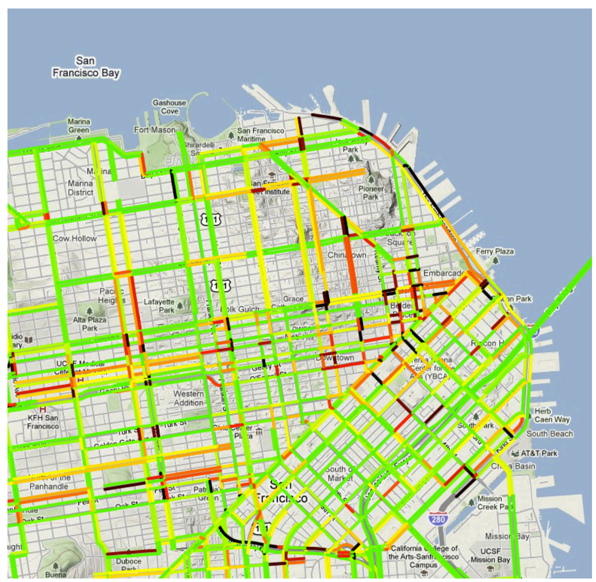
•iPhone application DriveTracker for San Francisco commuters

- •Real-time traffic conditions from Bayesian network model of individual link travel-times
- •2626 links, mean and variance of link travel-time available for 40 time periods during the day (up to 15-minute resolution)



•Communication scheme

- •Optimal policy is sent to the phone at origin
- •Policy updates pushed to phone if traffic conditions change significantly

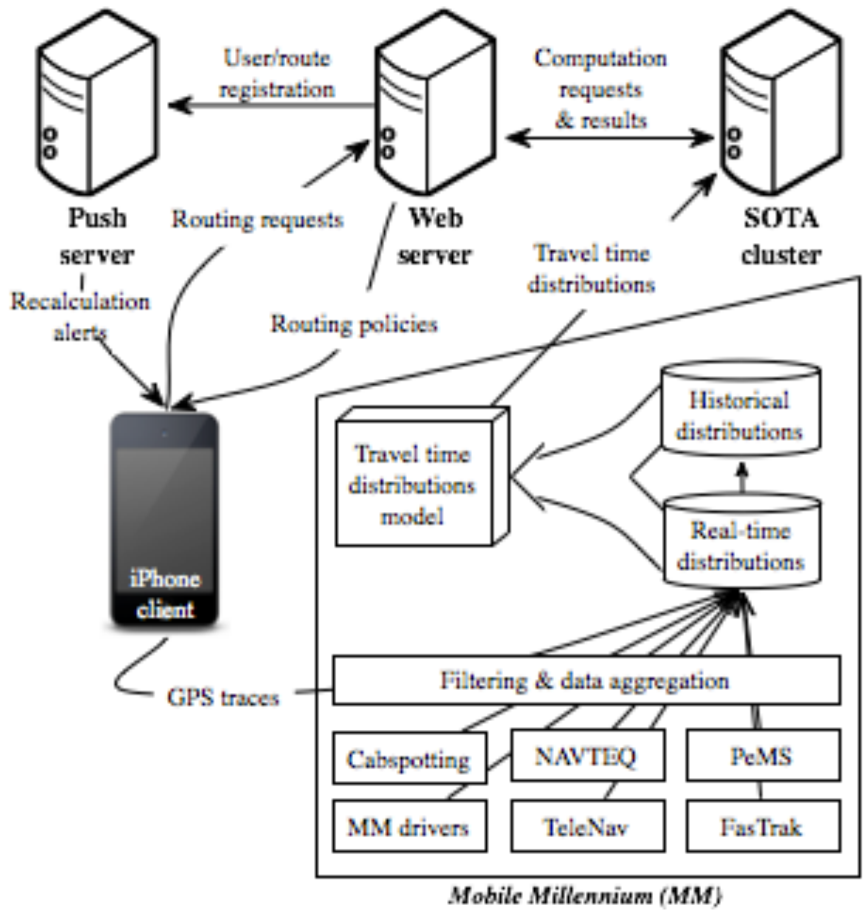


[Borokhov, Blandin, Samaranayake, Goldschmidt, Bayen, ITSC, 2011]

SOTA iPhone app features

- Power efficiency
  - Policy is computed only once
  - Push-based recomputation
  - Sampling scheme depends on dynamics

- Safety
  - Relative aural turn-by-turn directions
  - Minimal visual/cognitive distraction



[Borokhov, Blandin, Samaranayake, Goldschmidt, Bayen, ITSC, 2011]

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Network-wide optimal control

- All current control strategies are semi-local or microscopic
  - Ramp metering
  - Coordinated signal (corridor)
  - Route guidance

- Significant progress recently on traffic estimation
  - Truly distributed knowledge of traffic phenomena
  - Truly real-time estimation with high resolution
  - True knowledge of stochastic nature of traffic

- Optimal control of network traffic
  - Smart phones allow new types of controls: splitting rates at intersections not limited to {0,1}
  - Development of educated multimodality

- Better understanding of traffic dependencies and demand patterns to harness increased computational capabilities

Mobile Millennium project

http://traffic.berkeley.edu

Real-Time Integration of Lagrangian Sensors and Traffic Flow Models

Sébastien Blandin
Systems Engineering, UC Berkeley
blandin@berkeley.edu

HPC and cloud computing workshop
CITRIS, UC Berkeley
June 22nd, 2011

Real-Time Integration of Lagrangian Sensors and Traffic Flow Models

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Some candidate recursive estimation methods

particle filtering

Monte Carlo method

fully nonlinear, non-differentiable model and observation equation

extended Kalman filtering (not applicable)

requires linearized model (and observation) equation

storage of large covariance matrices

ensemble Kalman filtering

Monte Carlo model evolution, linear update equation

[Anderson and Moore, *Optimal Filtering*, 1979; Kaipio and Somersalo, *Statistical and Computational Inverse Problems* 2004; Evensen, *Data Assimilation: The Ensemble Kalman Filter*, 2006]

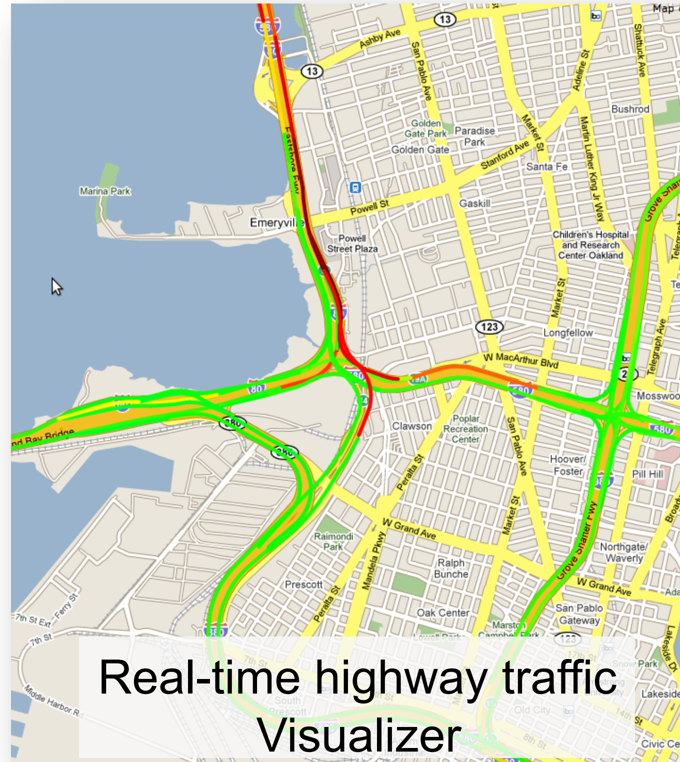
Network traffic estimation in *Mobile Millennium*

- directed graph representation
  - generated from Navteq map database (automated)
  - deployable nationwide

- Northern California network
  - 4164 edges, state dimension: >15,000
  - 3639 vertices (custom LP solver)

- production code
  - streaming data to participating users' cell phones for 18+ months

- real-time on my 4 year old



Real-time highway traffic Visualizer

[Work, Blandin, Tossavainen, Piccoli, Bayen, 2009]

Bayesian network for traffic estimation and forecast

Model assumptions:

- Jointly Gaussian
- Independent contemporary states given their parents

Historical data

Structure and parameters learning

Graph structure, parameters with maximal score (e.g. BIC score ~ MLE) on historical data

Current data

Forecast

Posterior forecast given model assumptions, current data, graph structure, parameters

Today

Modeling	LWR PDE	v-PDE on network	Phase transition PDE
Estimation		Ensemble Kalman Filter	
Control	Lyapunov boundary stabilization	Stochastic adaptive routing	

Motivation

•Highway traffic phenomena can be measured via a wide variety of sensors

•Fixed sensors
- •Reporting macroscopic quantities (count, occupancy)
- •Reporting microscopic quantities (travel time, speed)

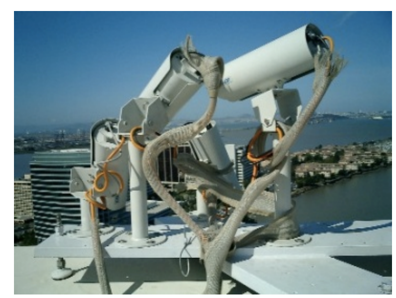
•Floating sensors
- •Probe vehicles from fleets (reporting traces)
- •Commuters using smart phone applications (speed)

•Mobile Millennium: testbed for novel traffic algorithms
- •60 million data points collected daily
- •Fusion platform providing real-time traffic in Bay Area









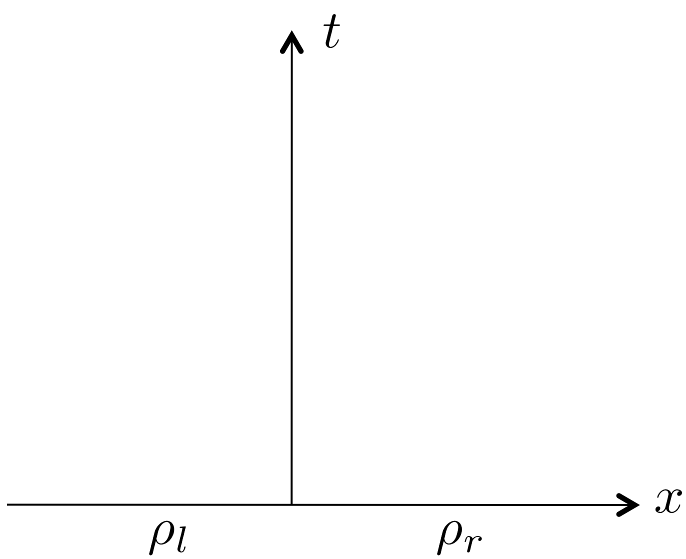
Riemann problem: building block for conservation laws

- Knowledge of dynamics and left and right constant initial condition

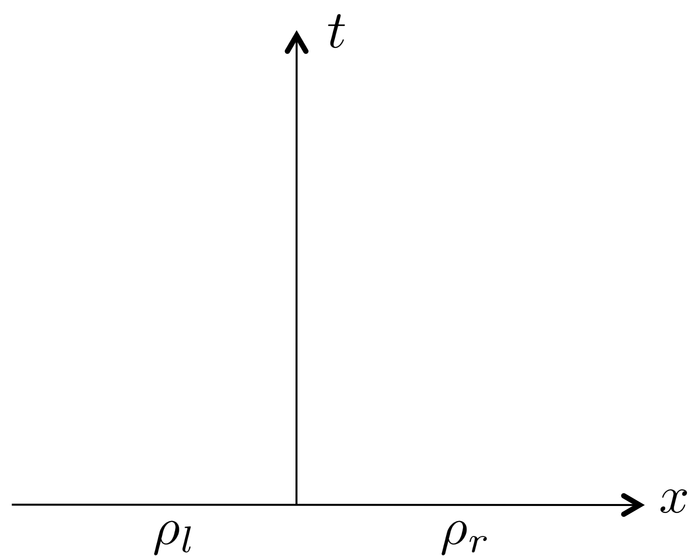
$$\begin{cases} \partial_t \rho + \partial_x Q(\rho) = 0 \\ \rho(0,x) = \begin{cases} \rho_l & \text{if} \quad x < 0 \\ \rho_r & \text{if} \quad x > 0 \end{cases} \end{cases}$$

- Traffic state ρ propagates at characteristic speed $Q'(\rho)$
- Two cases:

$Q'(\rho_l) \leq Q'(\rho_r)$



$Q'(\rho_l) > Q'(\rho_r)$



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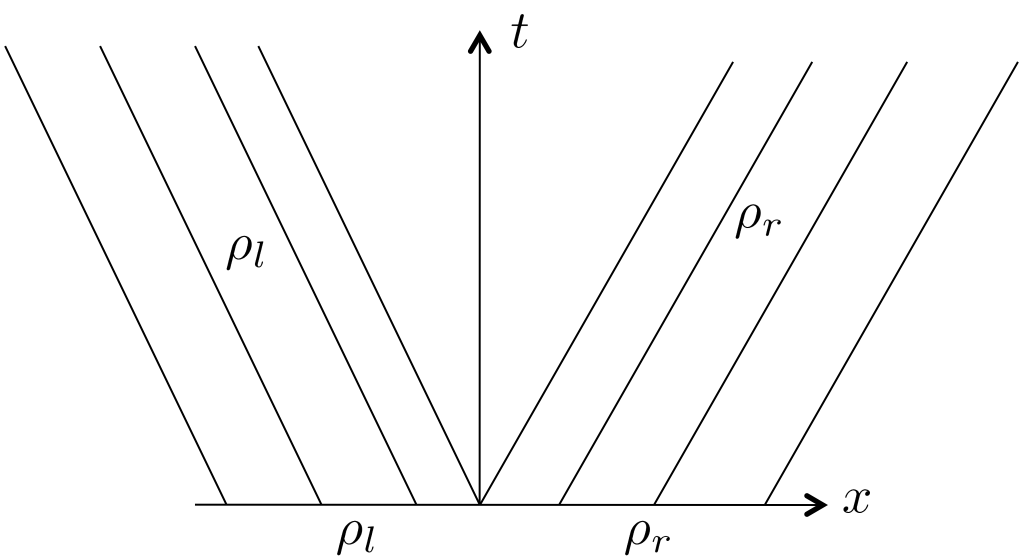

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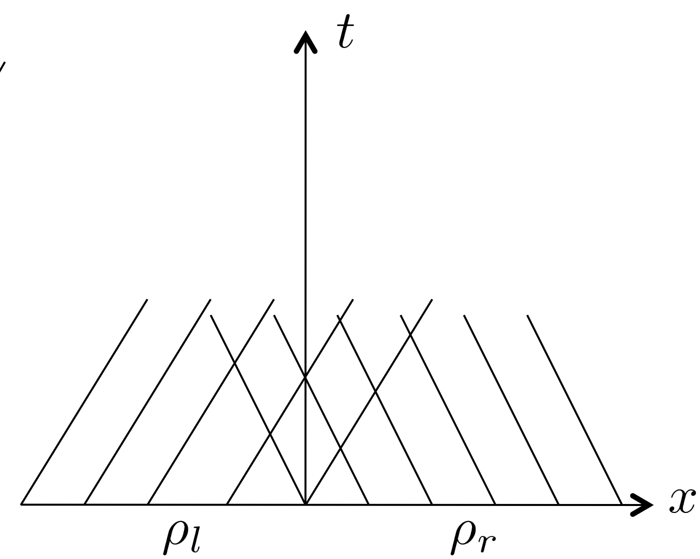

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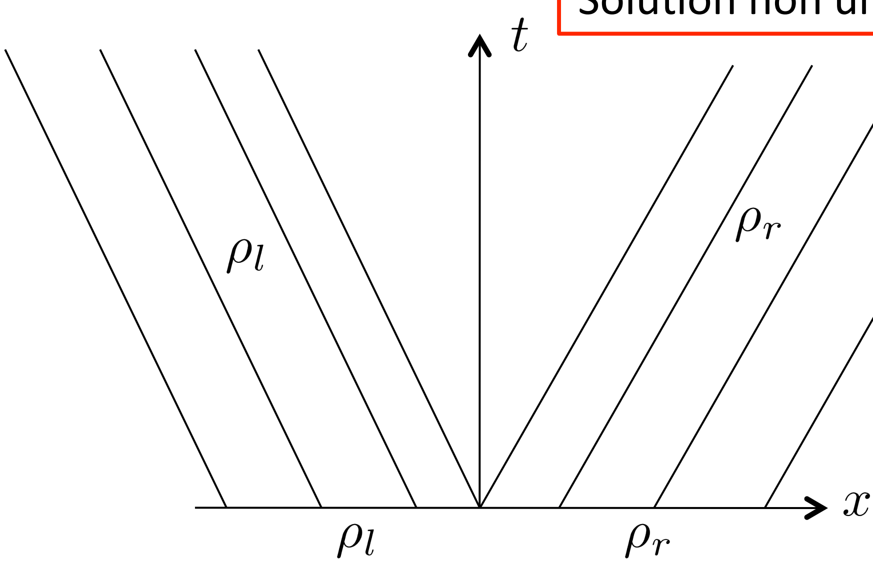
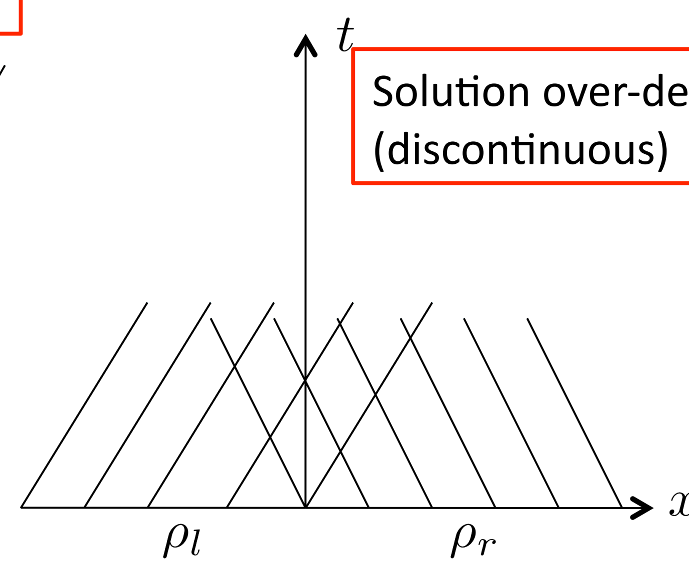
- Traffic state ρ propagates at characteristic speed $Q'(\rho)$
- Two cases:

$$Q'(\rho_l) \leq Q'(\rho_r) \qquad\qquad Q'(\rho_l) > Q'(\rho_r)$$

Solution non unique



Solution over-defined (discontinuous)



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- Two cases:

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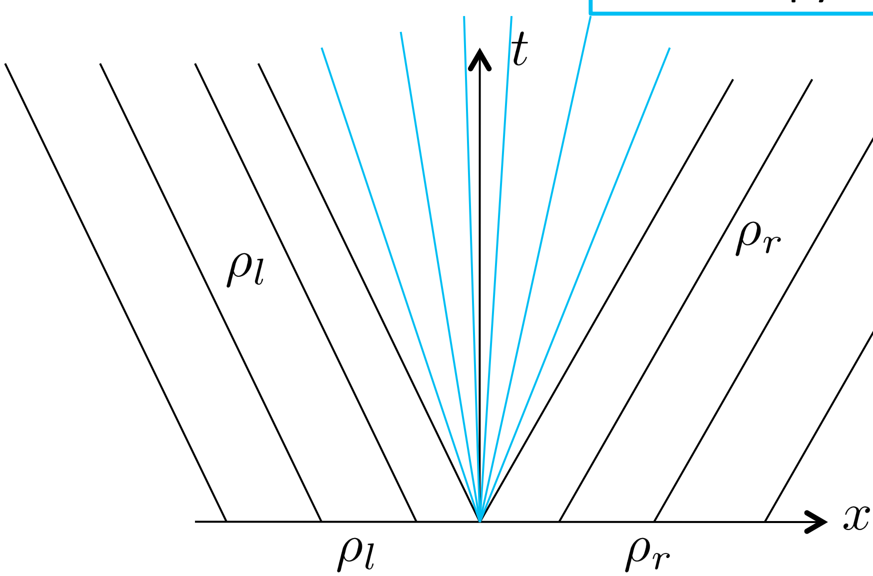
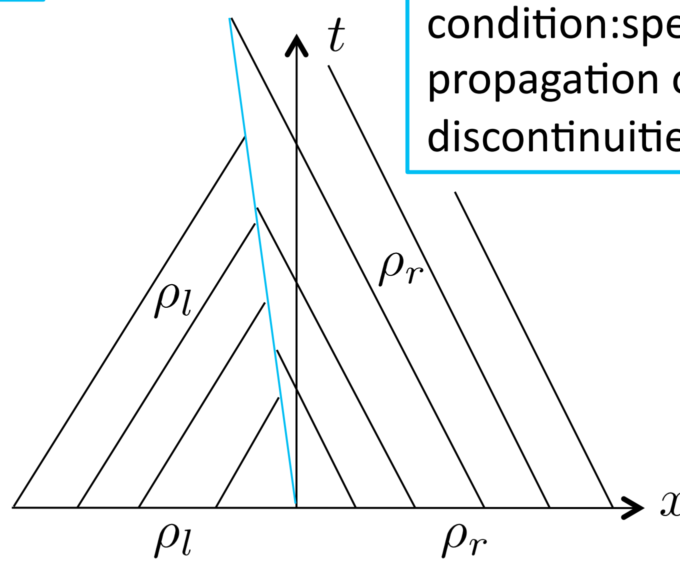
Lax entropy condition



$Q'(\rho_l) > Q'(\rho_r)$

Rankine-Hugoniot condition:speed of propagation of discontinuities



Lyapunov stability analysis: derivations

- Derivative of the Lyapunov function:

$$\frac{dV}{dt}(t) = \frac{1}{2}\left[\tilde{u}_-^2(t, x_1(t))\,\frac{dx_1}{dt} + \int_a^{x_1(t)} \partial_t \tilde{u}^2\,dx\right]$$
$$+\frac{1}{2}\sum_{i=1}^{N(t)-1}\left[\tilde{u}_-^2(t, x_{i+1}(t))\,\frac{dx_{i+1}}{dt} - \tilde{u}_+^2(t, x_i(t))\,\frac{dx_i}{dt}\right] + \frac{1}{2}\sum_{i=1}^{N(t)-1}\int_{x_i(t)}^{x_{i+1}(t)} \partial_t \tilde{u}^2\,dx$$
$$+\frac{1}{2}\left[-\tilde{u}_+^2(t, x_{N(t)}(t))\,\frac{dx_{N(t)}}{dt} + \int_{x_{N(t)}(t)}^{b} \partial_t \tilde{u}^2\,dx\right]$$

Where $u_\pm(t,x) = \lim_{h\to 0} u(t, x \pm h)$

- Using the fact that u is a solution of Burgers equation, and Rankine-Hugoniot relation:

$$\frac{dV}{dt}(t) = \frac{1}{3}u^3(t,a) - \frac{1}{2}u^2(t,a)\,u^* - \frac{1}{3}u^3(t,b) + \frac{1}{2}u^2(t,b)\,u^* + \frac{1}{12}\sum_{i=1}^{N(t)}\left(\Delta_i u\right)^3$$

Lyapunov stability analysis: derivations

- Derivative of the Lyapunov function:

$$
\begin{aligned}
\frac{dV}{dt}(t) &= \frac{1}{2}\left[\tilde{u}_-^2(t, x_1(t))\,\frac{dx_1}{dt} + \int_a^{x_1(t)} \partial_t \tilde{u}^2\,dx\right] \\
&+ \frac{1}{2}\sum_{i=1}^{N(t)-1}\left[\tilde{u}_-^2(t, x_{i+1}(t))\,\frac{dx_{i+1}}{dt} - \tilde{u}_+^2(t, x_i(t))\,\frac{dx_i}{dt}\right] + \frac{1}{2}\sum_{i=1}^{N(t)-1}\int_{x_i(t)}^{x_{i+1}(t)} \partial_t \tilde{u}^2\,dx \\
&+ \frac{1}{2}\left[-\tilde{u}_+^2(t, x_{N(t)}(t))\,\frac{dx_{N(t)}}{dt} + \int_{x_{N(t)}(t)}^{b} \partial_t \tilde{u}^2\,dx\right]
\end{aligned}
$$

Where $u_\pm(t,x) = \lim_{h\to 0} u(t, x \pm h)$

- Using the fact that u is a solution of Burgers equation, and Rankine-Hugoniot relation:

$$
\frac{dV}{dt}(t) = \underbrace{\frac{1}{3}u^3(t,a) - \frac{1}{2}u^2(t,a)\,u^* - \frac{1}{3}u^3(t,b) + \frac{1}{2}u^2(t,b)\,u^*}_{\text{boundary term}} + \underbrace{\frac{1}{12}\sum_{i=1}^{N(t)}\left(\Delta_i u\right)^3}_{\text{internal term}}
$$

where $\Delta_i u = u_+(t, x_i(t)) - u_-(t, x_i(t))$

Lyapunov stability analysis: remarks

$$\frac{dV}{dt}(t) = \underbrace{\frac{1}{3}u^3(t,a) - \frac{1}{2}u^2(t,a)\,u^* - \frac{1}{3}u^3(t,b) + \frac{1}{2}u^2(t,b)\,u^*}_{\text{boundary term}} + \underbrace{\frac{1}{12}\sum_{i=1}^{N(t)}(\Delta_i u)^3}_{\text{internal term}}$$

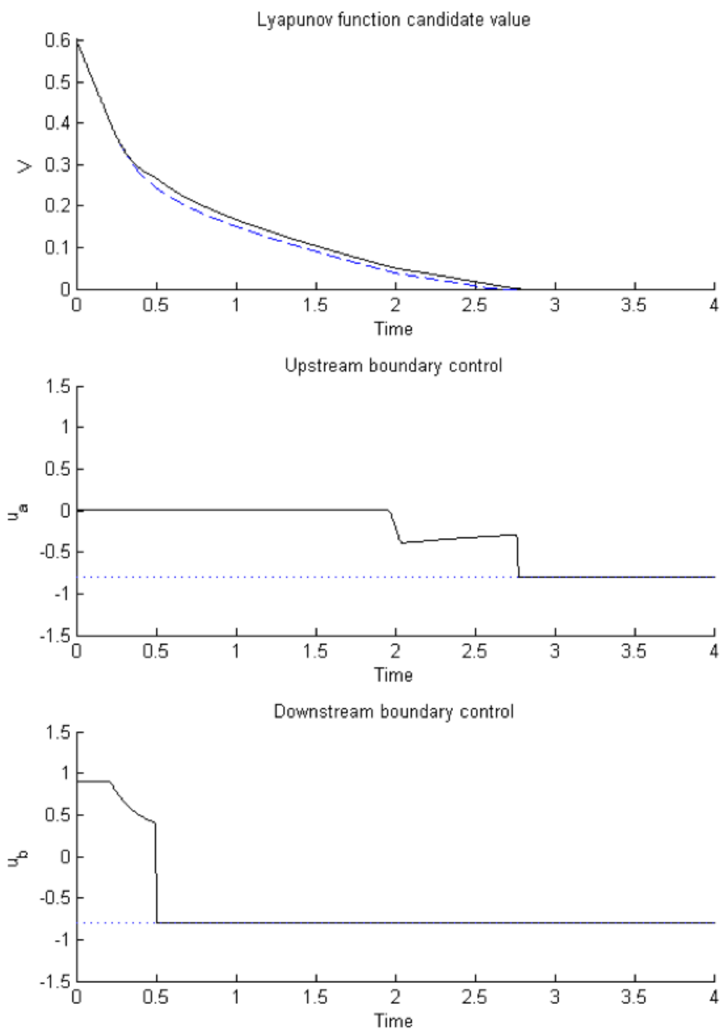
where $\Delta_i u = u_+(t, x_i(t)) - u_-(t, x_i(t))$ and $u_\pm(t,x) = \lim_{h\to 0} u(t, x \pm h)$

- According to Lax entropy condition, the internal term is stable

- The Lyapunov function is decreasing if $f(u(t,a)) - f(u(t,b)) \leq 0$
where $f : u \mapsto u^3/3 - u^* u^2/2$

- The value taken by the solution at the boundary may differ from the value imposed at the boundary

Numerical illustration



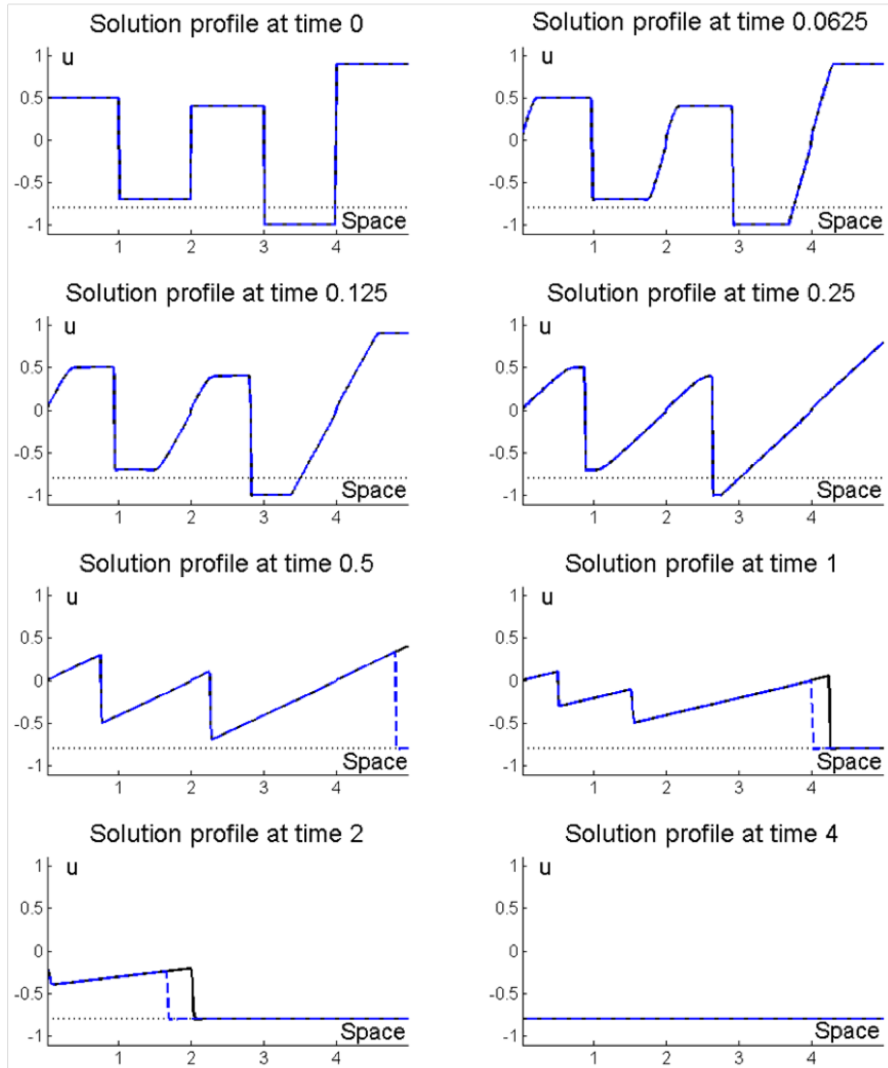
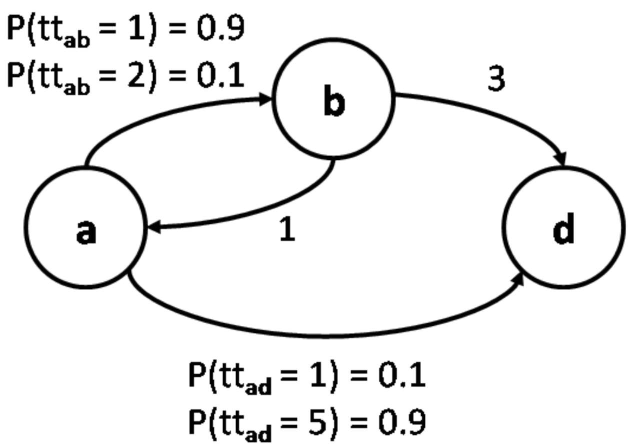


Illustration of loops in SOTA algorithm

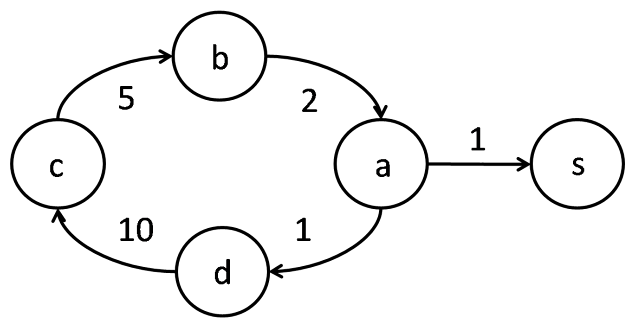
•Time budget of 4

•Different paths with non-zero probability of arriving on time at d when starting from a



Path	Travel-time	Probability
$\{(a,b),(b,d)\}$	4	0.9
$\{(a,d)\}$	1	0.1
$\{(a,b),(b,a),(a,d)\}$	4	0.01

Optimal order for the SOTA algorithm



Iter.	a	b	c	d
1	1	3	8	18
2	19	21	26	36
3	37	39	44	54
4	55	57	62	72

Table 3: computing the u_i values in the order (a, b, c, d).

Iter.	d	c	b	a
1	10	5	2	11
2	15	7	13	16
3	17	12	18	18
4	22	23	20	23

Table 4: computing the u_i value in the order (d, c, b, a).

Problem statement

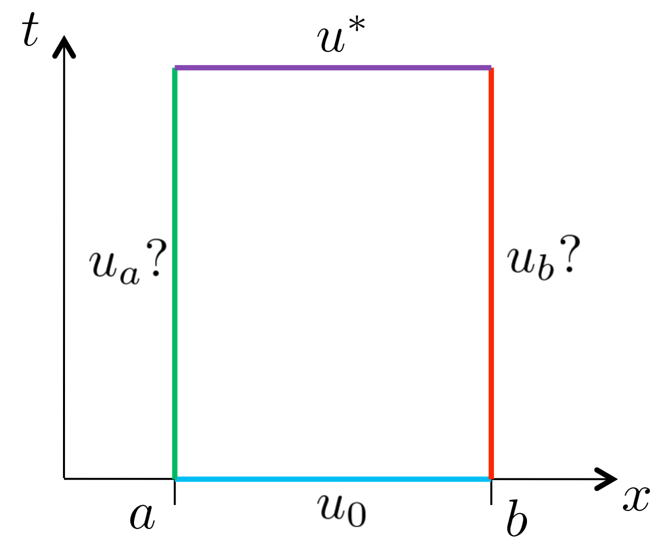
- Consider the Burgers partial differential equation:

$$\partial_t u + \tfrac{1}{2}\partial_x u^2 = 0$$

- Given
  - an initial condition $u_0(\cdot)$
  - a stationary state $u^*(\cdot)$

- Can one define left and right boundary conditions $u_a(\cdot), u_b(\cdot)$ such that the solution $u(\cdot,\cdot)$ of the initial-boundary value problem:

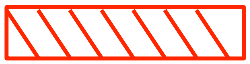
$$\begin{cases} \partial_t u + \frac{1}{2}\partial_x u^2 = 0 \\ u(0,x) = u_0(x) \\ u(a,t) = u_a(t), \quad u(b,t) = u_b(t) \end{cases}$$



is stable at $u^*(\cdot)$?

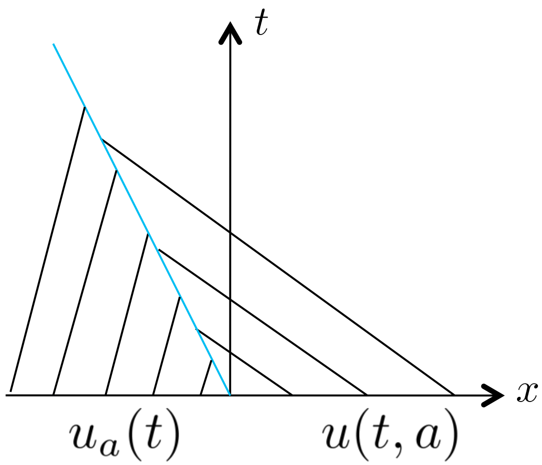
Scalar conservation laws: boundary control space

a.e. t



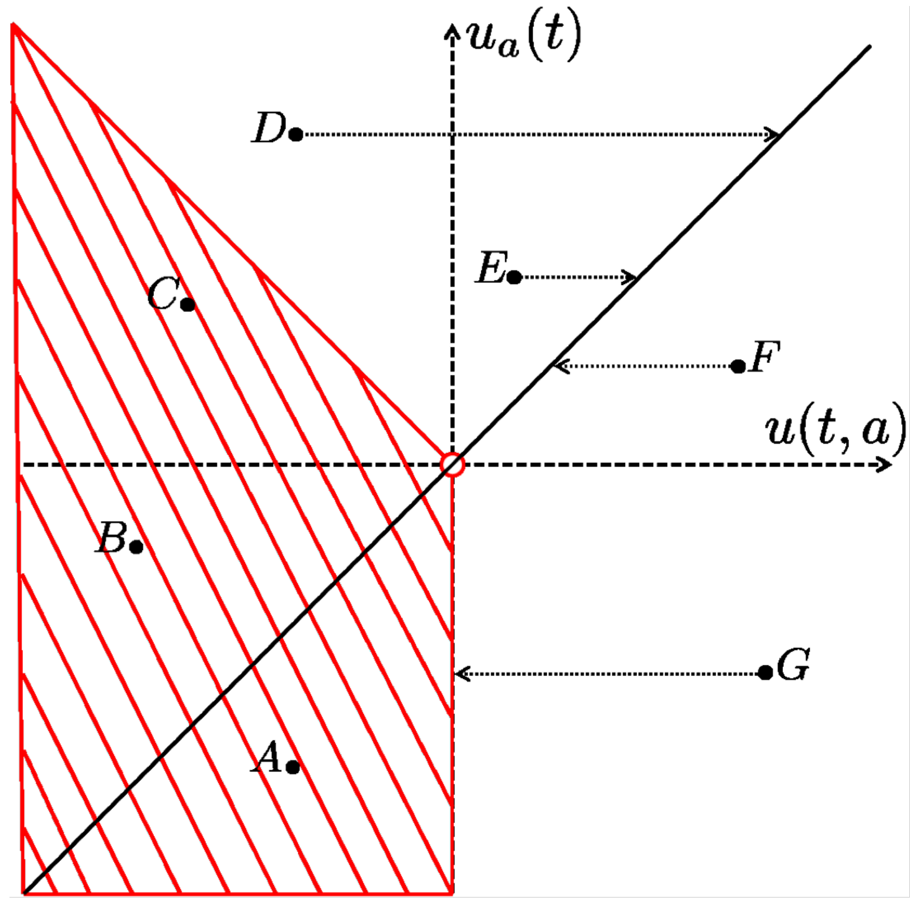
$$\begin{cases} u(t,a) = u_a(t) \quad \text{xor} \\ u(t,a) \leq 0 \text{ and } u_a(t) \leq 0 \text{ and } u(t,a) \neq u_a(t) \quad \text{xor} \\ u(t,a) \leq 0 \text{ and } u_a(t) > 0 \text{ and } \frac{1}{2}\,u^2(t,a) \geq \frac{1}{2}u_a^2(t) \end{cases}$$

- A, B, C: the control does not have any action.



Scenario C

- Example: imposing free-flow state from upstream with higher flow.



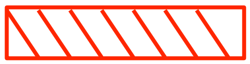
Upstream boundary

[Blandin, Litrico, Bayen, CDC, 2010]

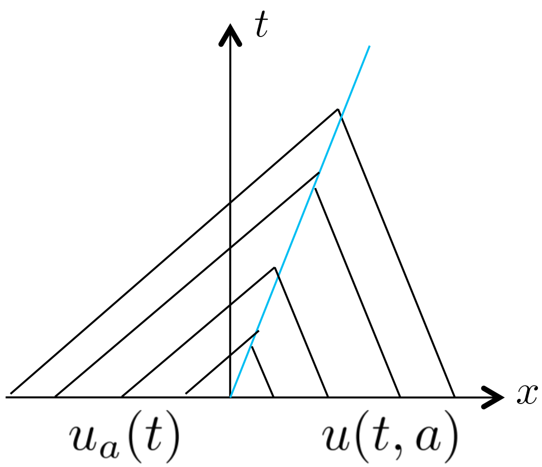
Scalar conservation laws: boundary control space

a.e. t

$$\begin{cases} u(t,a) = u_a(t) \quad \text{xor} \\ u(t,a) \leq 0 \text{ and } u_a(t) \leq 0 \text{ and } u(t,a) \neq u_a(t) \quad \text{xor} \\ u(t,a) \leq 0 \text{ and } u_a(t) > 0 \text{ and } \frac{1}{2}u^2(t,a) \geq \frac{1}{2}u_a^2(t) \end{cases}$$

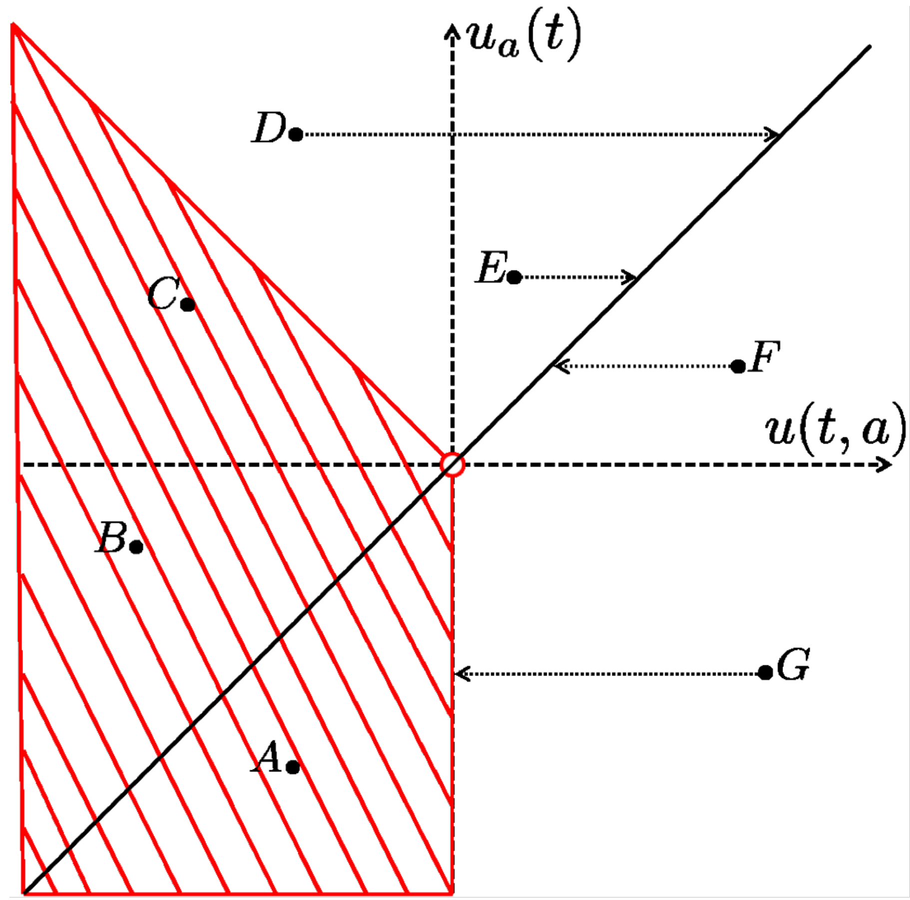


- D, E, F: the trace of the solution takes the value of the control



Scenario D

- Example: imposing free-flow state from upstream with lower flow.



Upstream boundary

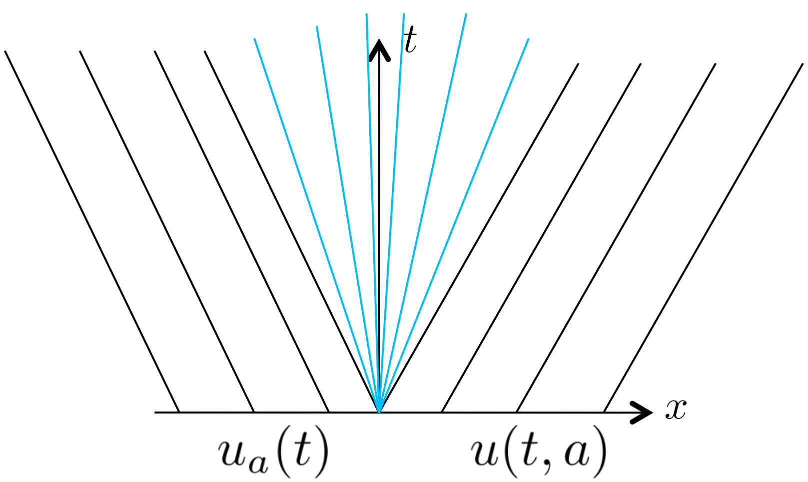
[Blandin, Litrico, Bayen, CDC, 2010]

Scalar conservation laws: boundary control space

a.e. t

$$\begin{cases} u(t,a) = u_a(t) \quad \text{xor} \\ u(t,a) \leq 0 \text{ and } u_a(t) \leq 0 \text{ and } u(t,a) \neq u_a(t) \quad \text{xor} \\ u(t,a) \leq 0 \text{ and } u_a(t) > 0 \text{ and } \frac{1}{2}\,u^2(t,a) \geq \frac{1}{2}u_a^2(t) \end{cases}$$

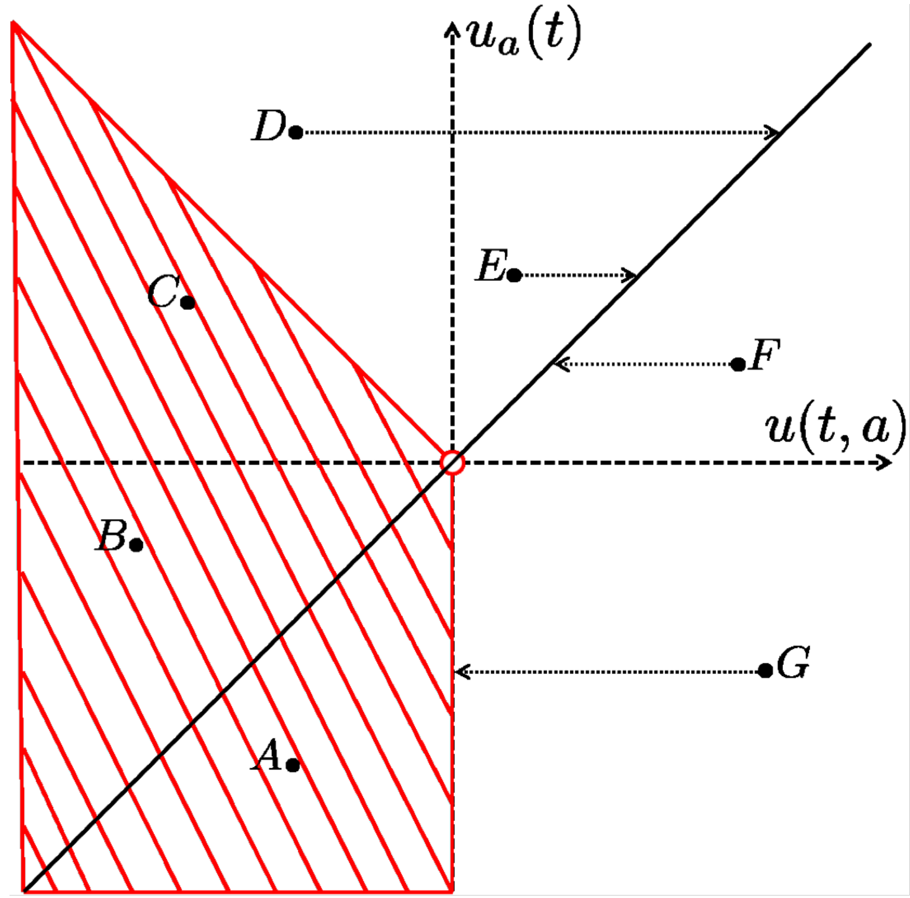
- G: the trace of the solution changes, but does not take the control value.



Scenario G

- Example: downstream end of a queue.

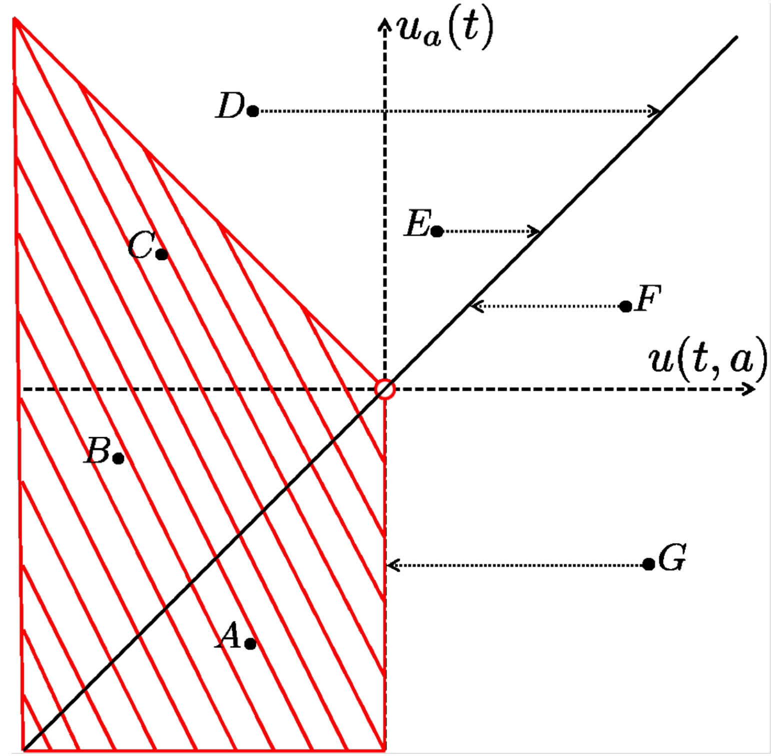
[Blandin, Litrico, Bayen, CDC, 2010]



Upstream boundary

Scalar conservation laws: boundary control space

- Different regions of the control space correspond to different types of controls, shockwave or rarefaction.



Upstream boundary

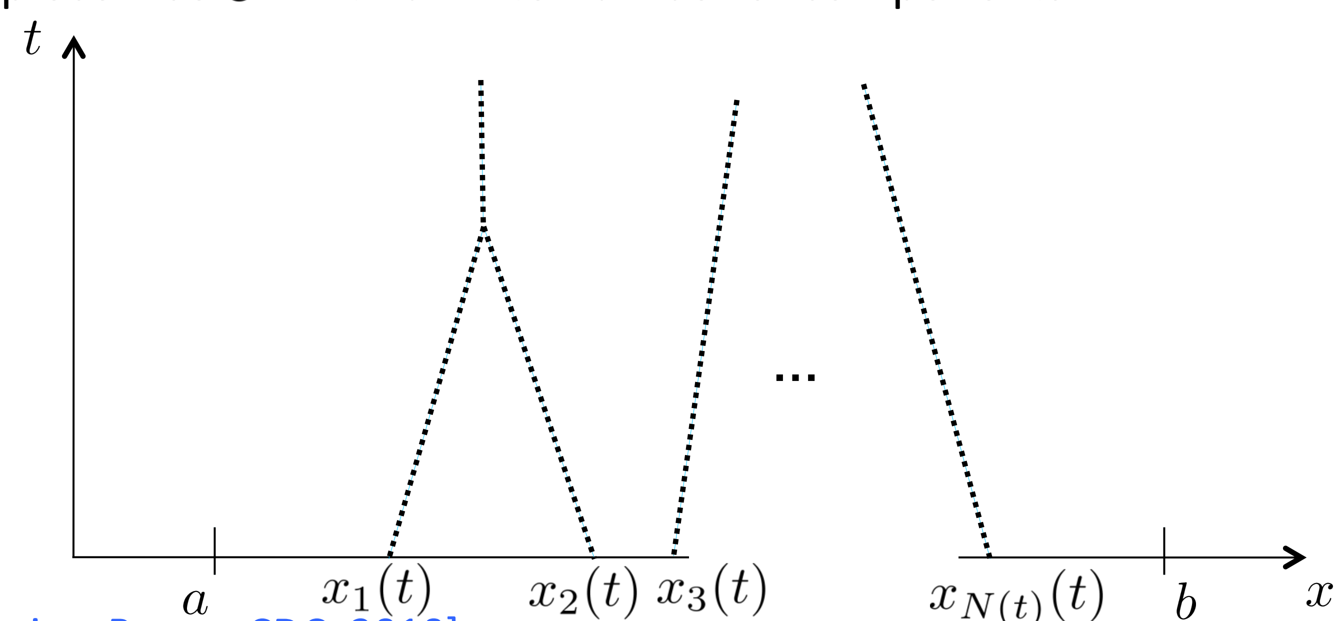
	$u(t,a) < 0$	$u(t,a) \geq 0$
$u_a(t) \geq 0$	$u_a(t) > -u(t,a)$ Shock	$u_a(t) > u(t,a)$: Shock $u_a(t) = u(t,a)$: No wave $u_a(t) < u(t,a)$: Rarefaction
$u_a(t) < 0$	$u_a(t) = u(t,a)$ No wave	$u_a(t) \in \emptyset$ Rarefaction with vanishing boundary trace

[Blandin, Litrico, Bayen, CDC, 2010]

Lyapunov stability analysis

- Notations:
  - $u(\cdot,\cdot)$ the solution to the initial-boundary value problem associated with the Burgers equation and the initial condition $u_0(\cdot)$ and the left and right boundary conditions $u_a(\cdot), u_b(\cdot)$
  - u^* the state at which we study the stability
  - $\tilde{u} = u - u^*$

- Assumption:
  - u is piecewise C^1 with a finite number of components



[Blandin, Litrico, Bayen, CDC, 2010]

Lyapunov stability analysis: differentiability

- Lyapunov function candidate

$$V(t) = \int_a^b \tilde{u}^2(t,x)dx$$

- Analysis
  1. Lyapunov candidate is well defined and differentiable
  2. Computation of derivative
  3. Leverage PDE solution properties to assess stabilizability

- The Lyapunov function can be re written as:

$$V(t) = \frac{1}{2}\int_a^{x_1(t)} \tilde{u}^2(t,x)\,dx + \frac{1}{2}\sum_{i=1}^{N(t)-1} \int_{x_i(t)}^{x_{i+1}(t)} \tilde{u}^2(t,x)\,dx + \frac{1}{2}\int_{x_{N(t)}(t)}^{b} \tilde{u}^2(t,x)\,dx$$

where:

- $x_i(\cdot)$ satisfies the Rankine-Hugoniot relation:

$$\frac{dx_i(t)}{dt} = \frac{1}{2}\left(u_+(t,x_i(t)) + u_-(t,x_i(t))\right)$$

- $\tilde{u}(t,\cdot)$ is continuously differentiable on $(x_i(t), x_{i+1}(t))$

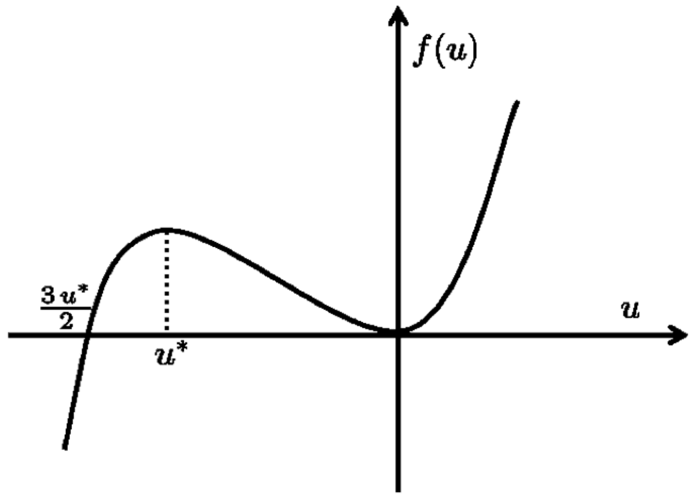
[Blandin, Litrico, Bayen, CDC, 2010]

Controller design: methodology

- Lyapunov derivative:

$$\frac{dV}{dt}(t) = \frac{1}{3}u^3(t,a) - \frac{1}{2}u^2(t,a)\,u^* - \frac{1}{3}u^3(t,b) + \frac{1}{2}u^2(t,b)\,u^* + \frac{1}{12}\sum_{i=1}^{N(t)}(\Delta_i u)^3$$

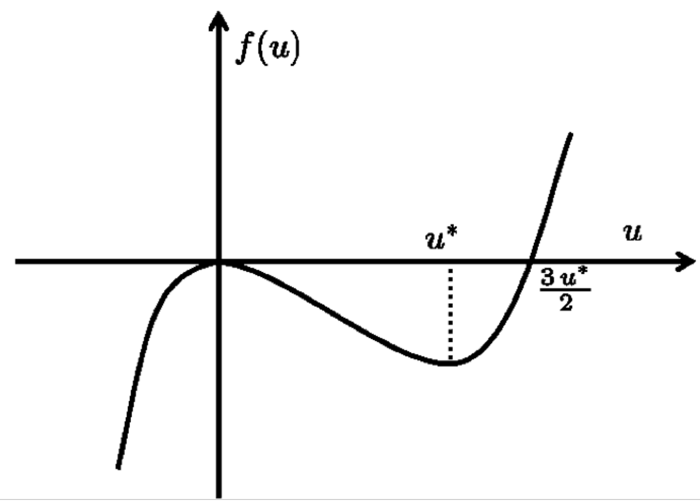
- With
  - $f : u \mapsto u^3/3 - u^* u^2/2$
  - $\mathcal{C}$ the control space



- Define the controller as the solution of:

$$u_a(t) = \arg \min_{\{u | (u(t,a),u) \in \mathcal{C}\}} f(u)$$

$$u_b(t) = \arg \max_{\{u | (u(t,b),u) \in \mathcal{C}\}} f(u)$$



[Blandin, Litrico, Bayen, CDC, 2010]

Controller design: stabilizability result

Theorem: For $u_0 \in BV(a,b)$, if the weak entropy solution $u(\cdot,\cdot)$ to the initial-boundary value problem can be written as a finite sum of continuously differentiable functions, then, under the boundary control:

$$u_a(t) := \begin{cases} \text{If } u^* \leq 0: \begin{cases} u(t,a) & \text{if } u(t,a) < 0 \\ 0 & \text{if } u(t,a) \geq 0 \end{cases} \\ \text{If } u^* > 0: \begin{cases} u^* & \text{if } u(t,a) \geq 0 \text{ OR}\big[\\ & \quad u(t,a) < 0 \text{ AND } u(t,a) > -u^* \text{ AND } f(u^*) \leq f(u(t,a)) \,\big] \\ u(t,a) & \text{if } u(t,a) < 0 \text{ AND}\big[\\ & \quad u(t,a) \leq -u^* \text{ OR}\big(u(t,a) > -u^* \text{ AND } f(u^*) > f(u(t,a))\,\big)\,\big] \end{cases} \end{cases}$$

$$u_b(t) := \begin{cases} \text{If } u^* \geq 0: \begin{cases} u(t,b) & \text{if } u(t,b) > 0 \\ 0 & \text{if } u(t,b) \leq 0 \end{cases} \\ \text{If } u^* < 0: \begin{cases} u^* & \text{if } u(t,b) \leq 0 \text{ OR}\big[\\ & \quad u(t,b) > 0 \text{ AND } u(t,b) < -u^* \text{ AND } f(u^*) \geq f(u(t,b)) \,\big] \\ u(t,b) & \text{if } u(t,b) > 0 \text{ AND}\big[\\ & \quad u(t,b) \geq -u^* \text{ OR}\big(u(t,b) < -u^* \text{ AND } f(u^*) < f(u(t,b))\,\big)\,\big] \end{cases} \end{cases}$$

the system is stable at u^*

[Blandin, Litrico, Bayen, CDC, 2010]